EXAM ~~1~~ 2
Math 4163
~~2-17~~-12
3-30

Name key

Instructions *Work all of the following problems in the space provided. If there is not enough room, you may write on the back sides of the pages. Give thorough explanations to receive full credit.*

You may use any formula that has been derived in the text or in class without having to rederive it.

1. *(30 points)* Solve Laplace's equation,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}=0,$$

for $u(r,\theta)$ on the half-disc $0<r<2$, $0<\theta<\pi$, with the boundary conditions

$$\frac{\partial u}{\partial \theta}(r,0)=0 \quad \text{and} \quad \frac{\partial u}{\partial \theta}(r,\pi)=0$$

for $0\le r\le 2$, and

$$u(2,\theta)=f(\theta)$$

for $0\le\theta\le\pi$, where $f(\theta)$ is a given function. You may assume that u is bounded near $r=0$.

Let $u(r,\theta)=G(r)h(\theta)$; Then from Laplace's equation we have

$\frac{1}{r}\frac{d}{dr}\left(r\frac{dG}{dr}\right)\cdot h(\theta)+\frac{1}{r^2}G(r)\cdot\frac{d^2h}{d\theta^2}=0$, so 2

$$\frac{r^2\cdot\frac{1}{r}\frac{d}{dr}\left(r\frac{dG}{dr}\right)}{G(r)}+\frac{\frac{d^2h}{d\theta^2}}{h(\theta)}=0, \quad \text{so} \quad \frac{r\left(r\frac{d^2G}{dr^2}+1\cdot\frac{dG}{dr}\right)}{G(r)}=\frac{-\frac{d^2h}{d\theta^2}}{h(\theta)}=\lambda$$ 2

for some constant λ. 2 Therefore $r^2\frac{d^2G}{dr^2}+r\frac{dG}{dr}-\lambda G=0$ and 2 $\frac{d^2h}{d\theta^2}=-\lambda h(\theta)$. The latter equation has boundary conditions $\frac{dh}{d\theta}(0)=0$ and $\frac{dh}{d\theta}(\pi)=0$, and we know the eigen~~values~~ functions for the h equation with 2 these BC's are $\cos(nx)$ 2 with eigenvalue $\lambda=n^2$ $(n=0,1,2,\dots)$

From class we know the equation $r^2\frac{dG}{dr^2}+r\frac{dG}{dr}-n^2G=0$ has 4 general solution $G=Cr^n+Dr^{-n}$ if $n>0$ or $G=C+D\ln r$ if $n=0$. Since $G(0)<\infty$, we have $G=Cr^n$ $(n>0)$ or $G=C$ $(n=0)$. 2

So $\boxed{u(r,\theta)=A_0+\sum_{n=1}^{\infty}A_nr^n\cos(n\theta)}$. 4

From $u(2,\theta)=f(\theta)$ we get $f(\theta)=A_0+\sum_{n=1}^{\infty}A_n2^n\cos(n\theta)$. 2

Therefore, by the formula for the coefficients in a Fourier cosine series on $0\le\theta\le\pi$, we have 2 $\boxed{A_0=\frac{1}{\pi}\int_0^{\pi}f(\theta)d\theta}$ and

$A_n2^n=\frac{2}{\pi}\int_0^{\pi}f(\theta)\cos(n\theta)d\theta$, 2 or 2 $\boxed{A_n=\frac{2}{2^n\pi}\int_0^{\pi}f(\theta)\cos(n\theta)d\theta}$ $(n=1,2,\dots)$

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2. *(15 points)* The function $f(x)$ is defined for $-\pi \le x \le \pi$ by

$$f(x) = \begin{cases} 0 & \text{for } -\pi < x < 0 \\ 1 & \text{for } 0 < x < \pi. \end{cases}$$

a) Find the coefficients of the Fourier series for $f(x)$.

$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos(nx) + B_n \sin(nx)$, where

(1) (1) (1)

$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_0^{\pi} 1\,dx = \frac{\pi}{2\pi} = \frac{1}{2}$

(1) (1) (1)

$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = \frac{1}{\pi}\int_0^{\pi}\cos(nx)\,dx = \frac{1}{\pi}\left[\frac{\sin(nx)}{n}\right]_0^{\pi} = \frac{1}{\pi n}[\sin(\pi n)] = 0$

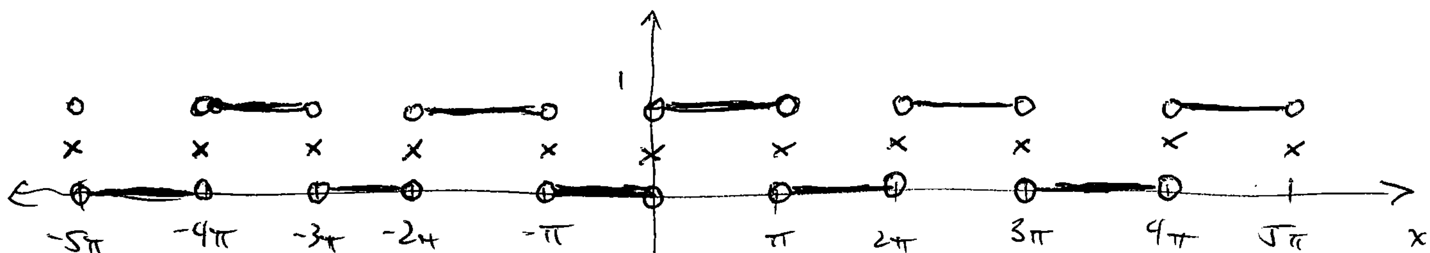
(1) (2) (2)

$B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \frac{1}{\pi}\int_0^{\pi}\sin(nx)\,dx = \frac{1}{\pi}\left[\frac{-\cos(nx)}{n}\right]_0^{\pi} = \frac{1}{\pi n}[1-\cos(n\pi)]$

$= \begin{cases} \frac{2}{\pi n} & (n \text{ odd}) \\ 0 & (n \text{ even}) \end{cases}$

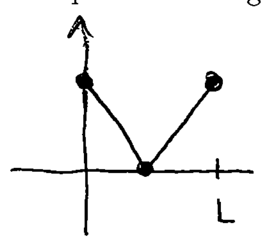
So $f(x) = \frac{1}{2} + \frac{2}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \dots\right)$

b) Sketch the graph of the function to which the Fourier series converges on $-5\pi \le x \le 5\pi$. Use x's to mark points showing what the Fourier series converges to at jump discontinuities.

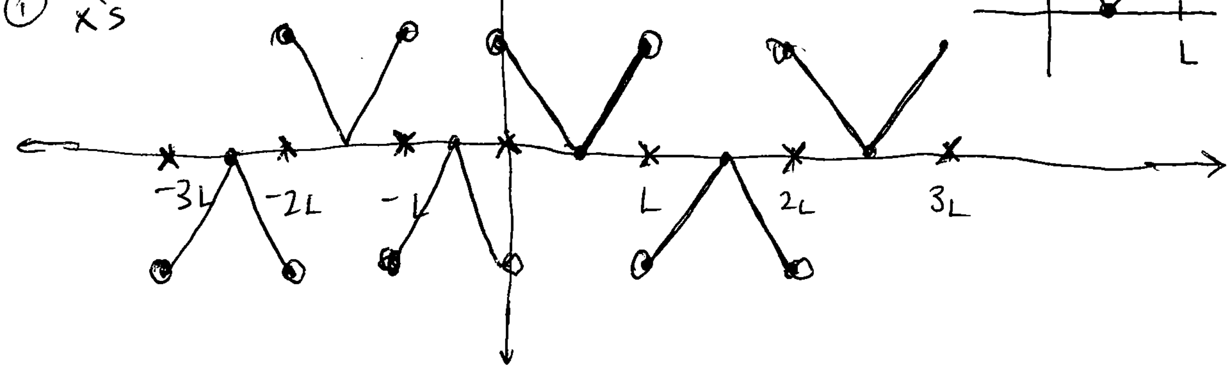


(2) odd
(2) periodic
(1) x's

3. *(5 points)* The graph of a function $g(x)$ defined for $0 \le x \le L$ is drawn in the diagram. Sketch the function to which the Fourier sine series of g converges on $-3L \le x \le 3L$. Use x's to mark points showing what the Fourier cosine series converges to at jump discontinuities.



(2) odd
(2) periodic
(1) x's



4. *(25 points)* Solve the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2},$$

for $u(x,t)$ on $0 < x < 1$, $t > 0$, with the boundary conditions

$$u(0,t) = 0 \quad \text{and} \quad u(1,t) = 0$$

for $t > 0$, and the initial conditions

$$u(x,0) = 0 \quad \text{and} \quad \frac{\partial u}{\partial t}(x,0) = 1$$

for $0 \le x \le 1$.

Putting $u(x,t) = \varphi(x)G(t)$ into the wave equation gives, after separating variables, $\frac{G''(t)}{G(t)} = \frac{\varphi''(x)}{\varphi(x)} = -\lambda$, so

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$G''(t) = -\lambda G(t)$ and $\varphi''(x) = -\lambda\varphi(x)$. Also the boundary conditions give $\varphi(0)=0$ and $\varphi(1)=0$. 2 This means the ~~nontrivial solutions for~~ eigenfunctions $\varphi(x)$ are $\sin(n\pi x)$, with corresponding eigenvalues $\lambda = n^2\pi^2$ $(n=1,2,3,\dots)$ 2 Then $G''(t) = -n^2\pi^2 G(t)$,

so $G(t) = A\cos n\pi t + B\sin n\pi t$ 2 So

$$u(x,t) = \sum_{n=1}^{\infty} \sin n\pi x\,(A_n\cos n\pi t + B_n \sin n\pi t) \quad 2$$

From the initial conditions we get (since $\frac{du}{dt} = \sum_{n=1}^{\infty} \sin n\pi x(-A_n n\pi \sin n\pi t + n\pi B_n \cos n\pi t)$ 2

$$0 = \sum_{n=1}^{\infty} A_n \sin n\pi x \quad 2$$

and $$1 = \sum_{n=1}^{\infty} B_n \cdot n\pi \cdot \sin n\pi x. \quad 2$$

From the first equation we get $A_n = \frac{2}{1}\int_0^1 0 \sin n\pi x\,dx = 0$ 2

and from the second equation we get 2

$$n\pi \cdot B_n = \frac{2}{1}\int_0^1 1\cdot\sin n\pi x\,dx = \frac{2}{n\pi}\left[-\cos n\pi x\right]_0^1 = \frac{2(1-\cos n\pi)}{n\pi}.$$

So $$B_n = \frac{2(1-\cos n\pi)}{n^2\pi^2} = \begin{cases} \frac{4}{n^2\pi^2} & (n \text{ odd}) \\ 0 & (n \text{ even}) \end{cases}.$$ 4 So

$$u(x,t) = \sum_{n=1}^{\infty} \frac{2(1-\cos n\pi)}{n^2\pi^2}\sin(n\pi x)\sin(n\pi t) = \frac{4}{\pi^2}\left(\sin\pi x\sin\pi t + \frac{\sin 3\pi x}{3}\sin 3\pi t + \frac{\sin 5\pi x}{5}\sin 5\pi t + \dots\right)$$

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5. *(25 points)* Consider the Sturm-Liouville problem

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0,$$

for $0 < x < 1$, with boundary conditions

$$\phi(0) - \frac{d\phi}{dx}(0) = 0 \quad \text{and} \quad \phi(1) = 0.$$

a) Find an equation for the square roots $\sqrt{\lambda}$ of the eigenvalues (you can assume that all the eigenvalues are positive). Sketch a graph on which the first few values of $\sqrt{\lambda}$ are marked as coordinates of intersection points of curves.

The general solution of $\frac{d^2\varphi}{dx^2} = -\lambda\varphi$ is $\varphi(x) = A\cos\sqrt{\lambda}x + B\sin(\sqrt{\lambda}x)$, (2)

so $\frac{d\varphi}{dx} = A\sqrt{\lambda}(-\sin\sqrt{\lambda}x) + B\sqrt{\lambda}\cos(\sqrt{\lambda}x)$ (2). Then

$0 = \varphi(0) - \frac{d\varphi}{dx}(0) = A$ (2) $- B\sqrt{\lambda}$, so $A = B\sqrt{\lambda}$ (2); and

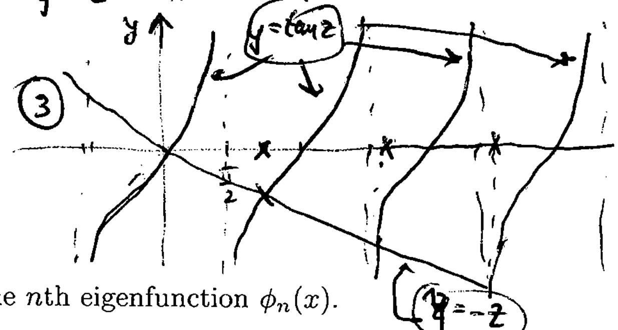
$0 = \varphi(1) = A\cos\sqrt{\lambda} + B\sin\sqrt{\lambda}$ (2). Since $A = B\sqrt{\lambda}$ then

$0 = B\sqrt{\lambda}\cos\sqrt{\lambda} + B\sin\sqrt{\lambda} = B(\sqrt{\lambda}\cos\sqrt{\lambda} + \sin\sqrt{\lambda})$ (1)

For non-trivial solutions, we must have $B \neq 0$ (because if $B = 0$ then $A = B\sqrt{\lambda} = 0$ also, so $\varphi \equiv 0$). Therefore $\sqrt{\lambda}\cos\sqrt{\lambda} + \sin\sqrt{\lambda} = 0$, (2)

so $\frac{\sin\sqrt{\lambda}}{\cos\sqrt{\lambda}} = -\sqrt{\lambda}$, or $\tan\sqrt{\lambda} = -\sqrt{\lambda}$. (2) Letting $z = \sqrt{\lambda}$ we see that z solves the equation $\tan z = -z$

so $\sqrt{\lambda}$ is the coordinate of an intersection point of the graphs $y = \tan z$ and $y = -z$ (see diagram →) (3)



b) Using λ_n to denote the nth eigenvalue, write a formula for the nth eigenfunction $\phi_n(x)$.

(2) $\varphi_n(x) = A\cos\sqrt{\lambda_n}x + B\sin\sqrt{\lambda_n}x$

$= B\sqrt{\lambda_n}\cos(\sqrt{\lambda_n}x) + B\sin(\sqrt{\lambda_n}x) = B(\sqrt{\lambda_n}\cos(\sqrt{\lambda_n}x) + \sin(\sqrt{\lambda_n}x))$.

c) How many zeros does $\phi_n(x)$ have in the interval $(0, L)$?

(2) From Sturm-Liouville Theory we know the n^{th} eigenfunction has $(n-1)$ zeros in $(0, L)$.

d) If $m \neq n$, what is the value of $\int_0^L \phi_n(x)\phi_m(x)\,dx$? Why?

(2) From Sturm-Liouville Theory, the eigenfunctions are orthogonal, satisfying $\int_0^L \varphi_n(x)\varphi_m(x)\,dx = 0$ for $m \neq n$.