EXAM 3 Name pey
Math 4163
4-30-12

Instructions *Work all of the following problems in the space provided. If there is not enough room, you may write on the back sides of the pages. Give thorough explanations to receive full credit.*

You may use any formula that has been derived in the text or in class without having to rederive it.

1. *(25 points)* Solve the wave equation
$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$
in a rectangular region $0 < x < L$, $0 < y < H$, with boundary conditions
$$u(0,y,t) = 0 \quad \text{and } u(L,y,t) = 0 \text{ for } 0 \le y \le H$$
and
$$\frac{\partial u}{\partial y}(x,0,t) = 0 \quad \text{and } \frac{\partial u}{\partial y}(x,H,t) = 0 \text{ for } 0 \le x \le L,$$
and initial conditions
$$u(x,y,0) = f(x,y) \quad \text{and } \frac{\partial u}{\partial t}(x,y,0) = 0.$$

Let $u(x,y,t) = \varphi(x,y)\,h(t)$, ~~Then~~ $\dfrac{\frac{d^2h}{dt^2}}{c^2 h} = \dfrac{\left(\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2}\right)}{\varphi} = -\lambda$, and

~~for Dirichlet boundary condts~~ so $\frac{d^2h}{dt^2} = -\lambda c^2 h$ and $\varphi_{xx} + \varphi_{yy} = -\lambda\varphi$.

Let $\varphi(x,y) = F(x)G(y)$; Then $\frac{F''(x)}{F(x)} + \frac{G''(y)}{G(y)} = -\lambda$ and

$\frac{F''(x)}{F(x)} + \lambda = \frac{-G''(y)}{G(y)} = \mu$, so $F''(x) = -(\lambda-\mu)F(x)$ and $G''(y) = -\mu G(y)$

The boundary conditions give $G'(0) = G'(H) = 0$, So $\mu = \left(\frac{m\pi}{H}\right)^2$ for $m = 0,1,2,$

and $G(y) = \cos\left(\frac{m\pi y}{H}\right)$ 5 Also $F(0) = F(L) = 0$, so $\lambda - \mu = \left(\frac{n\pi}{L}\right)^2$ for $n = 1,2,3,\ldots$

and $F(x) = \sin\left(\frac{n\pi x}{L}\right)$. 5 Therefore $\lambda = \lambda_{mn} = \left(\frac{m\pi}{H}\right)^2 + \left(\frac{n\pi}{L}\right)^2$, and

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$\frac{d^2h}{dt^2} = -\lambda_{mn} c^2 h \Rightarrow h(t) = A_{mn}\cos(c\sqrt{\lambda_{mn}}\,t) + B_{mn}\sin(c\sqrt{\lambda_{mn}}\,t)$.

~~Then~~ Since we want $\frac{du}{dt} = 0$ at $t = 0$, we can take $h'(0) = 0$, or 2 $B_{mn} = 0$

Hence $u(x,y,t) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} A_{mn}\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi y}{H}\right)\cos(c\sqrt{\lambda_{mn}}\,t)$ 3

Then $f(x,y) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} A_{mn}\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi y}{H}\right)$ and so

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$$A_{mn} = \frac{\int_0^L\int_0^H f(x,y)\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi y}{H}\right)dx\,dy}{\int_0^L\int_0^H \sin^2\left(\frac{n\pi x}{L}\right)\cos^2\left(\frac{m\pi y}{H}\right)dx\,dy}.$$

The denominator is $\frac{LH}{4}$ unless $m = 0$, in which case it is $\frac{LH}{2}$.

2. *(25 points)* Consider the heat equation on a disc,

$$\frac{\partial u}{\partial t} = k\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right)$$

for $0 < r < a$ and $-\pi < \theta < \pi$, with boundary condition

$$u(a, \theta, t) = 0$$

and initial condition

$$u(r, \theta, 0) = r^3.$$

Solve the problem by separation of variables.

NOTICE: Since the function $u(r, \theta, 0) = r^3$ in the initial condition does not depend on θ, neither will $u(r, \theta, t)$. Therefore you can start by writing $u = f(r)h(t)$.

Separating variables gives $\dfrac{\frac{dh}{dt}}{kh} = \dfrac{\frac{1}{r}\left(\frac{d}{dr}\left(r\frac{df}{dr}\right) + \frac{1}{r^2}\cdot 0\right)}{f} = -\lambda$,

so $\dfrac{dh}{dt} = -\lambda k h$ and $\dfrac{d}{dr}\left(r\dfrac{df}{dr}\right) = -\lambda r f$, or $r\dfrac{d^2f}{dr^2} + \dfrac{df}{dr} + \lambda r f = 0$,

or $r^2\dfrac{d^2f}{dr^2} + r\dfrac{df}{dr} + \lambda r^2 f = 0$. When $z = \sqrt{\lambda}\, r$, this becomes

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$z^2 f''(z) + z f'(z) + z^2 f(z) = 0$. This is Bessel's equation with $m = 0$, so the general solution is $f(z) = C J_0(z) + D Y_0(z)$.

But since $u(0, \theta, t) < \infty$ then $f(0) < \infty$, so $f(z) = C J_0(z)$, or

$f(r) = C J_0(\sqrt{\lambda}\, r)$. 5

Now $u(a, \theta, t) = 0$ implies $J_0(\sqrt{\lambda}\, a) = 0$, so $\sqrt{\lambda}\, a$ is a zero of $J_0(z)$. We write $\sqrt{\lambda}\, a = z_{0n}$ $(n = 1, 2, 3, \ldots)$. So

$\lambda = \lambda_n = \left(\dfrac{z_{0n}}{a}\right)^2$ and $f(r) = C J_0(\sqrt{\lambda_n}\, r)$. Also $\dfrac{dh}{dt} = -\lambda_n k h$,

so $h(t) = C e^{-\lambda_n k t}$ and separated solns are $C J_0(\sqrt{\lambda_n}\, r) e^{-\lambda_n k t}$ 5

The solution of the initial-value problem is

$$\boxed{u(r, \theta, t) = \sum_{n=1}^{\infty} C_n J_0(\sqrt{\lambda_n}\, r) e^{-\lambda_n k t}},$$ 5

and $u(r, \theta, 0) = r^3 \Rightarrow r^3 = \sum_{n=1}^{\infty} C_n J_0(\sqrt{\lambda_n}\, r)$ 5

$$\Rightarrow C_n = \frac{\int_0^a J_0(\sqrt{\lambda_n}\, r)\cdot r^3 \cdot r\, dr}{\int_0^a J_0(\sqrt{\lambda_n}\, r)\, r\, dr} = \boxed{\frac{\int_0^a J_0(\sqrt{\lambda_n}\, r)\, r^4\, dr}{\int_0^a J_0(\sqrt{\lambda_n}\, r)\, r\, dr}}$$

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3. *(25 points)* Consider the inhomogeneous heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 5x^2 t,$$

for $0 < x < 3$, with boundary conditions

$$u(0,t) = t^2 \quad \text{and} \quad \frac{\partial u}{\partial x}(3,t) = 12,$$

and initial condition

$$u(x,0) = x.$$

a) Find a simple function $v(x,t)$ so that v satisfies the given boundary conditions.

Take $v(x,t) = P(t) + x\,Q(t)$ 5

Then $\frac{\partial v}{\partial x}(x,t) = Q(t)$

So $v(0,t) = P(t) = t^2$ and $\frac{\partial v}{\partial x}(3,t) = Q(t) = 12$

so $v(x,t) = t^2 + 12x$ 5

b) Let $u(x,t) = v(x,t) + w(x,t)$, where v is the function you found in part a) above. Write down the problem that $w(x,t)$ satisfies, giving the differential equation, the boundary conditions and the initial condition. NOTICE: you are not being asked to solve the problem for w, just to write it down!

$v_t + w_t = v_{xx} + w_{xx} + 5x^2 t$

$\Rightarrow 2t + w_t = 0 + w_{xx} + 5x^2 t$

$\Rightarrow \boxed{w_t = w_{xx} + 5x^2 t - 2t}$ 5

Also $\boxed{w(0,t) = 0 \text{ and } \frac{\partial w}{\partial x}(3,t) = 0}$ 5

and $v(x,0) + w(x,0) = x$

$\Rightarrow 12x + w(x,0) = x$

$\Rightarrow \boxed{w(x,0) = -11x}$ 5

4. *(25 points)* Use the method of eigenfunction expansions to solve the inhomogeneous heat equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + 7x$$

for $0 < x < \pi$, with boundary conditions

$$u(0,t) = 0 \quad \text{and} \quad u(\pi,t) = 0,$$

and initial condition

$$u(x,0) = x.$$

You may use the fact that $x = \sum_{n=1}^{\infty} B_n \sin(nx)$ for $0 < x < \pi$, with $B_n = \dfrac{2(-1)^{n+1}}{\pi n}$.

~~u(x,t) =~~ The eigenfns. of $\frac{d^2\varphi}{dx^2} = -\lambda\varphi$ on $[0,\pi]$ w. Dirichlet BC's are $\varphi(x) = \sin(nx)$ $(n=1,2,\dots)$

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So put $\boxed{u(x,t) = \sum_{n=1}^{\infty} a_n(t)\sin(nx)}$.

Then $u_t = \sum \frac{da_n}{dt}\sin(nx)$ and $ku_{xx} = \sum_{n=1}^{\infty} k(-n^2)a_n(t)\sin(nx)$

and $7x = \sum 7B_n \sin(nx)$, so

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$$\sum_{n=1}^{\infty}\left(\frac{da_n}{dt} + kn^2 a_n - 7B_n\right)\sin(nx) = 0$$

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So $\frac{da_n}{dt} + kn^2 a_n - 7B_n = 0$, or $\frac{d}{dt}\left[e^{kn^2t}a_n\right] = 7B_n e^{kn^2t}$

so $e^{kn^2t}a_n\Big]_0^t = \left[\frac{7B_n e^{kn^2t}}{kn^2}\right]_0^t$ 2

or $e^{kn^2t}a_n(t) - 1\cdot a_n(0) = \frac{7B_n}{kn^2}e^{kn^2t} - \frac{7B_n}{kn^2}$ 2

or $a_n(t) = e^{-kn^2t}a_n(0) + \frac{7B_n}{kn^2}\left(1 - e^{-kn^2t}\right)$ 2

Since $u(x,0) = x$, then $x = \sum a_n(0)\sin nx \Rightarrow a_n(0) = B_n$.

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So $\boxed{a_n(t) = B_n e^{-kn^2t} + \frac{7B_n}{kn^2}\left(1 - e^{-kn^2t}\right)}$