FINAL EXAM Name *key*

Math 4163

5-11-12

<u>**Instructions**</u> *There are 6 problems for a total of 200 points. Work all of the following problems in the space provided. If there is not enough room, you may write on the back sides of the pages. Give thorough explanations to receive full credit.*

You may use any formula that has been derived in the text or in class without having to rederive it.

1. *(40 points)* Solve by separation of variables:

$$\frac{\partial u}{\partial t} = 3\left(\frac{\partial^2 u}{\partial x^2} + 5u\right) \quad \text{for } 0 < x < \pi,\ t > 0$$

with boundary conditions

$$u(0,t) = 0 \quad \text{and } u(\pi,t) = 0 \text{ for } t > 0$$

and initial condition

$$u(x,0) = f(x) \quad \text{for } 0 \le x \le \pi.$$

Let $u(x,t) = g(x)\cdot h(t)$, then $g \cdot \frac{dh}{dt} = 3\left(\frac{d^2g}{dx^2}\cdot h + 5gh\right)$, so 5

$\frac{\left(\frac{dh}{dt}\right)}{3h} = \frac{\left(\frac{d^2g}{dx^2}\right)}{g} + 5$, or $\frac{\left(\frac{dh}{dt}\right)}{3h} - 5 = \frac{\frac{d^2g}{dx^2}}{g} = -\lambda$. 5

The boundary conditions give $g(0) = g(\pi) = 0$, so nontrivial solutions of $\frac{d^2g}{dx^2} = -\lambda g$ exist for $n = 1, 2, 3, \cdots$ and equal $g(x) = \sin(nx)$. Then 5 5

$\lambda_n = n^2$ 5

$\frac{\frac{dh}{dt}}{3h} - 5 = -n^2 \Rightarrow \frac{dh}{dt} = 3(5-n^2)h \Rightarrow h(t) = C e^{3(5-n^2)t}$ 5

So separated solutions are $u(x,t) = C_n \sin(nx)\, e^{3(5-n^2)t}$,

and we look for a solution of the form 5

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} C_n \sin(nx)\, e^{3(5-n^2)t}}$$

Putting $t=0$ gives $f(x) = \sum_{n=1}^{\infty} C_n \sin(nx)$,

so $$\boxed{C_n = \frac{2}{\pi}\int_0^{\pi} f(y)\sin(ny)\,dy \quad (n = 1, 2, 3, \cdots)}$$ 5

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2. *(30 points)* Solve Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{for } 0 < x < L,\ 0 < y < H$$

with boundary conditions

$$\frac{\partial u}{\partial x}(0, y) = 0 \quad \text{and } u(L, y) = g(y) \text{ for } 0 \le y \le H$$

and

$$\frac{\partial u}{\partial y}(x, 0) = 0 \quad \text{and } \frac{\partial u}{\partial y}(x, H) = 0 \text{ for } 0 \le x \le L.$$

Putting $u(x,y) = g(x)\cdot h(y)$ we get $\frac{g''(x)}{g(x)} = -\frac{h''(y)}{h(y)}$ and 4

$g'(0)=0$, $h'(0)=0$, $h'(H)=0$. So take $\frac{g''}{g} = -\frac{h''}{h} = +\lambda$,

Then $h''(y) = -\lambda h(y)$ and the boundary conditions for h give

$\lambda = \left(\frac{n\pi}{H}\right)^2 \quad (n=0,1,2,\dots)$ and $h(y) = \cos\left(\frac{n\pi y}{H}\right)$. 4

Hence $\frac{d^2 g}{dx^2} = \left(\frac{n\pi}{H}\right)^2 g(x)$ which has general solution 4

$g(x) = A\cosh\left(\frac{n\pi x}{H}\right) + B\sinh\left(\frac{n\pi x}{H}\right)$. So

$g'(x) = A\left(\frac{n\pi}{H}\right)\sinh\left(\frac{n\pi x}{H}\right) + B\left(\frac{n\pi}{H}\right)\cosh\left(\frac{n\pi x}{H}\right)$ and $g'(0) = B\left(\frac{n\pi}{H}\right)$.

So $g'(0)=0 \Rightarrow B=0 \Rightarrow g(x) = A\cosh\left(\frac{n\pi x}{H}\right)$. 4

Therefore we look for a solution of the form

$$\boxed{u(x,y) = \sum_{n=0}^{\infty} A_n \cosh\left(\frac{n\pi x}{H}\right)\cos\left(\frac{n\pi y}{H}\right).}$$ 4

Putting $x=L$ gives

$$g(y) = \sum_{n=0}^{\infty} A_n \cosh\left(\frac{n\pi L}{H}\right)\cos\left(\frac{n\pi y}{H}\right),$$ 2

so $\boxed{A_n \cosh\left(\frac{n\pi L}{H}\right) = \frac{2}{H}\int_0^H g(y)\cos\left(\frac{n\pi y}{H}\right)dy}$, 4 or 2

(for $n \neq 0$) $\boxed{A_n = \frac{2}{H\cosh\left(\frac{n\pi L}{H}\right)}\int_0^H g(y)\cos\left(\frac{n\pi y}{H}\right)dy}$

2 and for $n=0$ $\boxed{A_0 = \frac{1}{H\cosh\left(\frac{n\pi L}{H}\right)}\int_0^H g(y)\,dy = \frac{1}{H}\int_0^H g(y)\,dy}$

3. *(40 points)* The function $f(x)$ is defined for $0 \le x \le 2$ by

$$f(x) = \begin{cases} 0 & \text{for } 0 < x < 1 \\ 5 & \text{for } 1 < x < 2. \end{cases}$$

a) Find the coefficients of the Fourier sine series for $f(x)$.

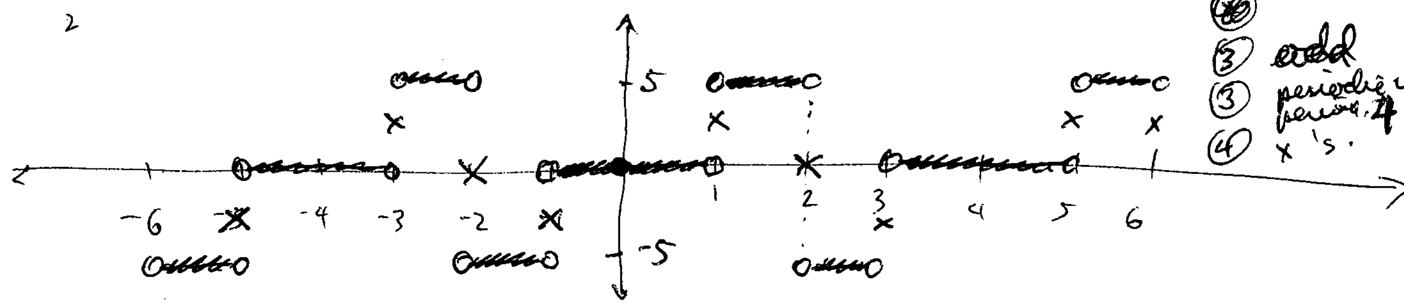
[10] $f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{2}\right)$ where $B_n = \frac{2}{2}\int_0^2 f(y)\sin\left(\frac{n\pi y}{2}\right)dy$ (2) (2)

$= \int_1^2 5\sin\left(\frac{n\pi y}{2}\right)dy$ (2) $= -\frac{5\cdot 2}{n\pi}\left[\cos\left(\frac{n\pi y}{2}\right)\right]_1^2$ (2)

$= \boxed{\frac{-10}{n\pi}\left[\cos(n\pi) - \cos\left(\frac{n\pi}{2}\right)\right]}$ (2)

b) Sketch the graph of the function to which the Fourier sine series converges on $-6 \le x \le 6$. Use x's to mark points showing what the Fourier sine series converges to at jump discontinuities.

[10]



(3) odd
(3) periodic w period 4
(4) x's.

c) Find the coefficients of the Fourier cosine series for $f(x)$.

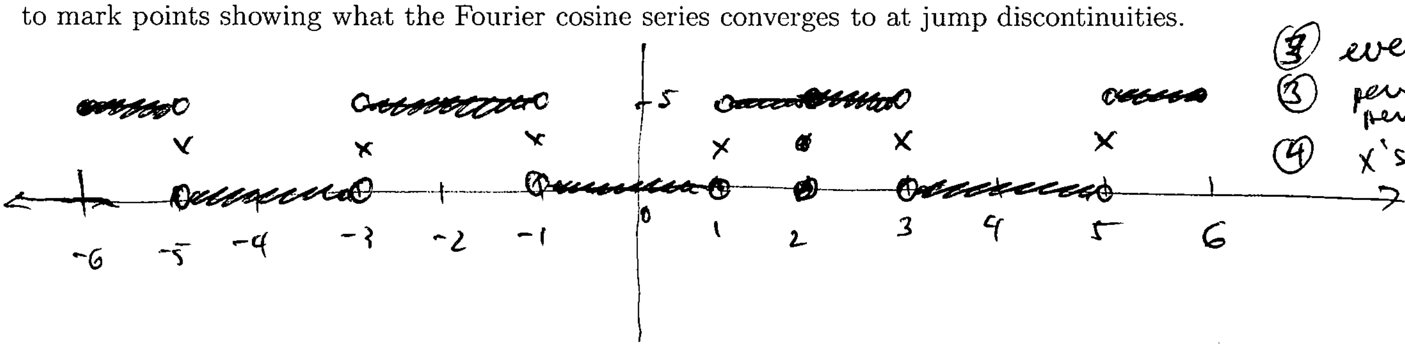
[10] $f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{2}\right)$ where $\begin{cases} A_n = \frac{2}{2}\int_0^2 f(y)\cos\left(\frac{n\pi y}{2}\right)dy & (n \ne 0) \\ A_0 = \frac{1}{2}\int_0^2 f(y)\,dy \end{cases}$ (2)

So for $n \ne 0$, $A_n = \int_1^2 5\cos\left(\frac{n\pi y}{2}\right)dy$ (2) $= \frac{10}{n\pi}\left[+\sin\left(\frac{n\pi y}{2}\right)\right]_1^2$ (2)

$= \frac{+10}{n\pi}\left[\sin(n\pi) - \sin\left(\frac{n\pi}{2}\right)\right]$ (2) $= \boxed{\frac{-10}{n\pi}\sin\left(\frac{n\pi}{2}\right)}$, and

for $n=0$, $A_0 = \frac{1}{2}\int_1^2 5\,dy = \frac{1}{2}\left[5y\right]_1^2 = \frac{1}{2}(10-5) =$ (2) $\boxed{\frac{5}{2}}$.

d) Sketch the graph of the function to which the Fourier cosine series converges on $-6 \le x \le 6$. Use x's to mark points showing what the Fourier cosine series converges to at jump discontinuities.



(3) even
(3) periodic period 4
(4) x's.

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4. *(30 points)* Use the method of eigenfunction expansions to solve the inhomogeneous heat equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + 5\sin(3x)$$

for $0 < x < \pi$, with boundary conditions

$$u(0,t) = 0 \quad \text{and} \quad u(\pi,t) = 0,$$

and initial condition

$$u(x,0) = 0.$$

(Hint: what is the Fourier sine series of $5\sin(3x)$?)

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We have $5\sin(3x) = \sum_{n=1}^{\infty} B_n \sin(nx)$ where $B_n = \begin{cases} 0 & n \neq 3 \\ 5 & n = 3 \end{cases}$.

Let $u(x,t) = \sum_{n=1}^{\infty} a_n(t)\sin(nx)$ 5 (since $\sin(nx)$ are the eigenfunctions for $\varphi'' = -\lambda\varphi$, $\varphi(0) = \varphi(\pi) = 0$.)

Then $\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} \frac{da_n}{dt}\sin(nx)$ and $\frac{\partial^2 u}{\partial x^2} = \sum_{n=1}^{\infty} (-n^2) a_n \sin(nx)$,

so the equation gives $\sum_{n=1}^{\infty} \frac{da_n}{dt}\sin(nx) = \sum_{n=1}^{\infty}(-k n^2 \cancel{\tfrac{da_n}{dt}})\sin(nx) + \sum_{n=1}^{\infty} B_n \sin(nx)$

or $\frac{da_n}{dt} + kn^2 a_n - B_n = 0$ 5 for $n = 1, 2, 3, \ldots$ The integrating factor is e^{kn^2t}, and multiplying by it gives $\frac{d}{dt}\left(a_n e^{kn^2t}\right) = B_n e^{kn^2t}$.

$\Rightarrow a_n e^{kn^2t} = \frac{B_n}{kn^2} e^{kn^2t} + C_n$ 5 $\Rightarrow a_n = \frac{B_n}{kn^2} + C_n e^{-kn^2t}$

Since $u(x,0) = 0$ then $a_n(0) = 0$, so $\frac{B_n}{kn^2} + C_n = 0$, so $C_n = \frac{-B_n}{kn^2}$ 5. For $n \neq 3$ we have $C_n = B_n = 0$ and for $n = 3$ we have $B_n = 5$ and $C_n = \frac{-5}{kn^2}$.

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So $u(x,t) = \sum_{n=1}^{\infty}\left(\frac{B_n}{kn^2} + C_n e^{-kn^2t}\right)\sin(nx) = \left[\frac{5}{9k} + \left(\frac{-5}{9k}\right)e^{-9kt}\right]\sin(3x)$

$$= \boxed{\frac{5}{9k}\left[1 - e^{-9kt}\right]\sin(3x)}$$

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5. *(30 points)* Find the solution of the wave equation on a disc for a function $u(r,t)$,

$$\frac{\partial^2 u}{\partial t^2} = k\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) \qquad (\text{put } k = c^2)$$

inside a circle of radius a, with boundary condition

$$u(a,t) = 0$$

and initial conditions

$$u(r,0) = 0$$

and

$$\frac{\partial u}{\partial t}(r,0) = g(r).$$

Write $u(r,t) = f(r)h(t)$, then $\dfrac{\left(\frac{d^2h}{dt^2}\right)}{c^2 h} = \dfrac{\frac{1}{r}\frac{d}{dr}\left(r\frac{df}{dr}\right)}{f} = -\lambda$, so 5

$\frac{d^2h}{dt^2} = -\lambda c^2 h$ and $\frac{d^2f}{dr^2} + \frac{1}{r}\frac{df}{dr} + \lambda f = 0$, or $r^2\frac{d^2f}{dr^2} + r\frac{df}{dr} + \lambda r^2 f = 0$.

Putting $z = \sqrt{\lambda}r$ gives $z^2\frac{d^2f}{dz^2} + z\frac{df}{dz} + z^2 f = 0$, 5 which is Bessel's equation of order 0, and has general solution $f(z) = CJ_0(z) + DY_0(z)$, or $f(r) = CJ_0(\sqrt{\lambda}r) + DY_0(\sqrt{\lambda}r)$. But since $u(0,t) < \infty$ then $f(0) < \infty$, so $D = 0$, and hence $f(r) = CJ_0(\sqrt{\lambda}r)$. 5 Also, $u(a,t) = 0$ $\Rightarrow f(a) = 0 \Rightarrow J_0(\sqrt{\lambda}a) = 0$. So $\sqrt{\lambda}a$ is one of the values of z for which $J_0(z) = 0$. If we write the zeros of $J_0(z)$ as $z_{01}, z_{02}, z_{03}, \ldots$, then $\sqrt{\lambda}a = z_{0n}$, so $\lambda = \lambda_n = \left(\frac{z_{0n}}{a}\right)^2$. 5

Therefore $\frac{d^2h}{dt^2} = -\lambda_n c^2 h \Rightarrow h(t) = A_n\cos(c\sqrt{\lambda_n}t) + B_n\sin(c\sqrt{\lambda_n}t)$

But $u(r,0) = 0 \Rightarrow h(0) = 0 \Rightarrow A_n = 0 \Rightarrow h(t) = B_n\sin(c\sqrt{\lambda_n}t)$,

so $\boxed{u(r,t) = \sum_{n=1}^{\infty} B_n J_0(\sqrt{\lambda_n}r)\sin(c\sqrt{\lambda_n}t).}$ 5

Then $\frac{du}{dt}(r,t) = \sum_{n=1}^{\infty} B_n J_0(\sqrt{\lambda_n}r)c\sqrt{\lambda_n}\cos(c\sqrt{\lambda_n}t)$, so

$g(r) = \sum_{n=1}^{\infty} B_n J_0(\sqrt{\lambda_n}r)c\sqrt{\lambda_n}$. Hence 5

$$\boxed{B_n = \frac{1}{c\sqrt{\lambda_n}}\left(\int_0^a g(r)J_0(\sqrt{\lambda_n}r)\cdot r\,dr\right)\Big/\left(\int_0^a J_0(\sqrt{\lambda_n}r)^2 r\,dr\right)}$$

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6. *(30 points)* Use Fourier transforms to solve the problem

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{for } -\infty < x < \infty,\ t > 0$$

with initial conditions

$$u(x, 0) = 0$$

and

$$\frac{\partial u}{\partial t}(x, 0) = g(x).$$

Your answer should express u as an integral involving the Fourier transform of g.

Let $\overline{U}(\omega,t)$ be the Fourier transform of $u(x,t)$,

$\overline{U}(\omega,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(x,t)\, e^{i\omega x}\,dx$. Then the Fourier transform of $\frac{\partial^2 u}{\partial t^2}$ is $\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial t^2} e^{i\omega x}\,dx = \frac{\partial^2}{\partial t^2}\frac{1}{2\pi}\int_{-\infty}^{\infty} u(x,t)\,e^{i\omega x}\,dx = \frac{\partial^2}{\partial t^2}\overline{U}$;

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and as we already know, the Fourier transform of $\frac{\partial^2 u}{\partial x^2}$ is $(-i\omega)^2\,\overline{U}(\omega,t) = -\omega^2\,\overline{U}(\omega,t)$. So taking the F. transform of the PDE gives (*)

$$\boxed{\frac{\partial^2 \overline{U}}{\partial t^2} = c^2(-\omega^2)\,\overline{U}.}$$ 5

Also taking the Fourier transform of the initial conditions gives

$\overline{U}(\omega,0) = 0$ and $\frac{\partial \overline{U}}{\partial t}(\omega,0) = G(\omega)$

where $G(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(x)\,e^{i\omega x}\,dx$ is the F. transform of g.

Fixing ω and solving (*) as an ODE in t, we get 5

$\overline{U}(\omega,t) = A(\omega)\cos(c\omega t) + B(\omega)\sin(c\omega t)$, and 5

$\overline{U}(\omega,0) = 0 \Rightarrow A(\omega) = 0 \Rightarrow \overline{U}(\omega,t) = B(\omega)\sin(c\omega t)$. Then

$\frac{d\overline{U}}{dt}(\omega,t) = B(\omega)\cdot c\omega\cos(c\omega t)$ so $G(\omega) = \frac{\partial \overline{U}}{\partial t}(\omega,0) = B(\omega)\,c\omega$, 5

and $B(\omega) = \frac{G(\omega)}{c\omega}$. So $\boxed{\overline{U}(\omega,t) = \frac{G(\omega)}{c\omega}\sin(c\omega t)}$ and 5

$$u(x,t) = \int_{-\infty}^{\infty} \overline{U}(\omega,t)\,e^{-i\omega x}\,d\omega = \boxed{\int_{-\infty}^{\infty} \frac{G(\omega)}{c\omega}\sin(c\omega t)\,e^{-i\omega x}\,d\omega}$$

(see next page)

#6 cont'd

[The solution to #6 can be carried a bit farther. It wasn't necessary to go beyond the formula at the bottom of the last page, but some of you did, so I'm including this bit for those of you who want to see if your extra work was correct:]

From the tables of Fourier transforms included with the exam, we see that $\frac{\sin(c\omega t)}{c\omega}$ is the Fourier transform of $\frac{\pi}{c}\cdot f(x)$, where $f(x) = \begin{cases} 0 & \text{if } |x| > ct \\ 1 & \text{if } |x| < ct \end{cases}$.

So by the convolution formula,

$$u(x,t) = \mathcal{F}^{-1}\left(G(\omega)\, \frac{\sin(c\omega t)}{c\omega} \right)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\pi}{c} f(y)\, g(x-y)\, dy$$

$$= \frac{1}{2c} \left[\int_{-\infty}^{-ct} \cancelto{0}{0\, g(x-y)\, dy} + \int_{-ct}^{ct} 1 \cdot g(x-y)\, dy + \int_{ct}^{\infty} \cancelto{0}{0 \cdot g(x-y)\, dy} \right]$$

so

$$\boxed{u(x,t) = \frac{1}{2c} \int_{-ct}^{ct} g(x-y)\, dy}$$