Quiz — Math 4163 — Feb. 15, 2012

Name: key

[10] **1.** Assuming $\lambda > 0$, find the eigenvalues and eigenfunctions for the boundary-value problem:

$$\frac{d^2\phi}{dx^2} = -\lambda\phi \qquad (0 < x < \pi),$$
$$\frac{d\phi}{dx}(0) = 0,$$
$$\phi(\pi) = 0.$$

The general solution of the ODE is $\varphi(x) = B_1 \cos(\sqrt{\lambda}x) + B_2 \sin(\sqrt{\lambda}x)$, (2)
whose derivative is $\frac{d\varphi}{dx} = B_1\sqrt{\lambda}(-\sin(\sqrt{\lambda}x)) + B_2\sqrt{\lambda}\cos(\sqrt{\lambda}x)$. (2)
From $\frac{d\varphi}{dx}(0) = 0$ we get $B_1\sqrt{\lambda}\cdot 0 + B_2\sqrt{\lambda}\cdot 1 = 0$, or $B_2 = 0$, so
$\varphi(x) = B_1\cos(\sqrt{\lambda}x)$. (2) From $\varphi(\pi) = 0$ we get $B_1\cos(\sqrt{\lambda}\pi) = 0$,
so $\sqrt{\lambda}\pi = \frac{n\pi}{2}$ where n is an odd integer. So $\sqrt{\lambda} = \frac{n}{2}$ (2) and $\boxed{\lambda = \frac{n^2}{4} \quad (n = 1, 3, 5, 7, \ldots)}$ are the eigenvalues, with corresponding eigenfunctions (2)
$\boxed{\varphi(x) = \cos\left(\frac{nx}{2}\right) \quad (n = 1, 3, 5, 7, \ldots)}$

[10] **2.** Give the solution of

$$\frac{\partial u}{\partial t} = 2\,\frac{\partial^2 u}{\partial x^2} \qquad (0 < x < \pi,\ t > 0),$$
$$\frac{\partial u}{\partial x}(0, t) = 0 \qquad (t \geq 0),$$
$$u(\pi, t) = 0 \qquad (t \geq 0),$$
$$u(x, 0) = \cos\frac{9x}{2} \qquad (0 \leq x \leq \pi).$$

(Note: the solution is of the form $u(x,t) = \phi(x)G(t)$.)

The PDE has separated solutions $u(x,t) = \varphi(x)G(t)$ (2)
where $\frac{d^2\varphi}{dx^2} = -\lambda\varphi$ and $\frac{dG}{dt} = -2\lambda G$. The boundary conditions imply that
~~from prob~~ $\frac{d\varphi}{dx}(0) = 0$ and $\varphi(\pi) = 0$, so from problem 1, above (1)
we get that $\lambda = \frac{n^2}{4}$ and $\varphi(x) = B\cos\left(\frac{nx}{2}\right)$, where n is $1, 3, 5, \ldots$
Also $G(t) = e^{-2\lambda t}$ (2) $= e^{-2\frac{n^2}{4}t} = e^{-(n^2/2)t}$ (2) Since
$u(x,0) = \varphi(x)\cdot G(0) = B\cos\left(\frac{nx}{2}\right)e^0$, ~~for~~ $u(x,0) = \cos\left(\frac{9x}{2}\right)$ we
should take $n = 9$. Then $u(x,t) = \varphi(x)G(t) = B\cos\left(\frac{nx}{2}\right)e^{-\frac{n^2}{2}t} = \boxed{\cos\left(\frac{9x}{2}\right)e^{-\frac{81}{2}t}}$ (2)
$B = 1$ (1)