**Math 4163 — Review for Exam 3**

Exam 3 covers sections 7.1, 7.2, 7.3, 7.4, 7.7, 8.1, 8.2, and 8.3 of the text. The relevant assignments are Assignments 6 and 7. Here is a summary of the material in these sections.

**7.1 Higher dimensional partial differential equations — Introduction.** Up till this chapter we had studied only equations for functions of two independent variables: the heat and wave equations for functions of the variables $x$ and $t$, and Laplace's equation for functions of the variables $x$ and $y$. Now we consider equations for functions of three independent variables: the heat and wave equations for functions of the variables $x$, $y$, and $t$. (In sections 7.9 and 7.10 three other equations are considered — Laplace's equation for functions of $x$, $y$, and $z$, and the heat and wave equations for functions $x$, $y$, $z$, and $t$. But that material is not treated in this class.)

**7.2 Separation of the time variable.** This section introduces the idea of a two-dimensional eigenvalue problems.

Till now, the only eigenvalue problems we had considered were one-dimensional problems, for functions $\phi$ of a single variable $x$. A typical example was the problem of finding eigenvalues and eigenfunctions for the equation

$$\frac{d^2\phi}{\partial x^2} = -\lambda\phi$$

on the interval $[0, L]$, with Dirichlet, or Neumann, or mixed boundary conditions on $\phi$.

In this chapter, however, we consider eigenvalue problems for functions $\phi$ of two variables $x$ and $y$. Namely, we consider the problem of finding eigenvalues and eigenfunctions for the equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = -\lambda\phi$$

on a two-dimensional region (in particular, on a rectangle or a disc), with Dirichlet, or Neumann, or mixed boundary conditions on $\phi$.

You should be aware that sometimes, as on page 279, the author uses the common notation $\nabla^2\phi$ as an abbreviation for the expression $\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}$. (It's also common to use $\Delta\phi$ to stand for the same thing, though this text doesn't do it.)

In the paragraph at the top of page 278, there seems to be the suggestion that the eigenvalues for this problem are necessarily positive. This is misleading: it is true that the eigenvalues are necessarily positive for Dirichlet boundary conditions, but not necessarily for Neumann or mixed boundary conditions. In particular, for the homework problem 7.7.2(a) on Assignment 6, $\lambda = 0$ is an eigenvalue. (See the discussion of section 7.7 below.)

**7.3 Vibrating rectangular membrane.** You should review the material in this section in detail, at least up through equation (7.3.27) on page 284.

The part of the section after (7.3.27) gives an explanation of how to find the coefficients $A_{nm}$ and $B_{nm}$ in the double Fourier series in terms of the initial data. The results are in equations (7.3.30) and (7.4.40). However, there's another way to obtain equation (7.3.30) and (7.4.40), which is explained briefly at the bottom of page 293 in section 7.4, and which I also explained in class.

**7.4. Eigenvalue problem $\nabla^2\phi + \lambda\phi = 0$.** This section doesn't include the solution of any new problems, but it's important for understanding how the rest of the material in the chapter fits together, and reading it will make it easier to use and remember all the formulas in the other sections.

The basic facts are in the box on page 290. Since we haven't covered Rayleigh quotients in this class, you can ignore statement number 6 in this box, but I'd recommend reading through the rest and carefully comparing these general statements with how they are borne out in the particular case of the rectangular membrane. This comparison is done for you on pages 291 to 293. As mentioned in the comments on section 7.3 above, understanding the orthogonality of the eigenfunctions (see equation (7.4.5) on page 290 ) makes it easy to see how to find the Fourier coefficients in the double Fourier series for the solution of the two-dimensional wave or heat equation.

**7.7. Vibrating circular membrane and Bessel functions.** First, this section describes how to separate variables for the wave equation on a disc. You should review the material from the beginning of the section through equations (7.7.18)-(7.7.20) on page 305.

What we are doing here is separating variables in polar coordinates for the equation

$$\nabla^2\phi + \lambda\phi = 0$$

on the disk. You may notice that the procedure is quite similar to the one we used earlier, in section 2.5.2, to solve the equation

$$\nabla^2\phi = 0$$

on the disc. In section 2.5.2, the eigenfunctions were found to be of the form $r^m \sin m\theta$ or $r^m \cos n\theta$, while here in section 7.7, the eigenfunctions are of the form $J_m(\sqrt{\lambda_{mn}}r)\sin m\theta$ or $J_m(\sqrt{\lambda_{mn}}r)\cos m\theta$. It would be a good idea to compare and contrast the derivations of the eigenfunctions in sections 2.5.2 and 7.7, to see what exactly is responsible for this difference. What role did the presence or absence of the $\lambda\phi$ term play?

What will you be expected to know about Bessel functions for the exam? Not too much. You should know Bessel's equation (7.7.23) and how it transforms into equation (7.7.25) via the change of variables $z = \sqrt{\lambda}r$. You should know that the general solution of Bessel's equation (7.7.25) is a linear combination of two types of Bessel function called $J_m(z)$ (Bessel functions of the first kind) and $Y_m(z)$ (Bessel functions of the second kind). You should have a rough idea of what these look like (see Figure 7.8.1 on page 319, or better yet go to http://www.wolframalpha.com/ and type "plot J_0(x)" or "plot BesselY[1,x]" into the orange box, and see what you get. In particular, you should know that only the Bessel functions of the first kind are bounded near $z = 0$, so we can exclude the Bessel functions of the second kind from our solutions. Apart from these facts, you don't need to know the rest of the material in subsections 7.7.5 or 7.7.6.

There is one other important thing you should know about Bessel functions. Since the eigenfunctions for the equation $\nabla^2\phi + \lambda\phi = 0$ on the disk are of the form $J_m(\sqrt{\lambda_{mn}}r)\sin m\theta$ or $J_m(\sqrt{\lambda_{mn}}r))\cos m\theta$, then the general facts about eigenfunctions listed on page 290 tell us that any two distinct eigenfunctions must be orthogonal. So we must have

$$\int\int_R J_m(\sqrt{\lambda_{mn}}r)\sin m\theta J_k(\sqrt{\lambda_{kl}}r)\sin k\theta \; dx \; dy = 0,$$

unless $m = k$ and $n = l$; and

$$\int\int_R J_m(\sqrt{\lambda_{mn}}r)\cos m\theta J_k(\sqrt{\lambda_{kl}}r)\cos k\theta \; dx \; dy = 0,$$

unless $m = k$ and $n = l$; and

$$\int\int_R J_m(\sqrt{\lambda_{mn}}r)\sin m\theta J_k(\sqrt{\lambda_{kl}}r)\cos k\theta \; dx \; dy = 0,$$

no matter whether $m = k$ and $n = l$ or not. You don't need to memorize these facts; they are just particular instances of the general formula in equation (7.4.5) on page 290.

Knowing this fact enables you to find the coefficients in the series expansion for the solution of the wave or heat equation on the disc, in the usual way, as integrals involving the initial data. This is explained, rather sketchily, at the bottom of page 312. A more detailed explanation is given in subsection 7.7.9. However, you should be aware that subsection 7.7.9 is dealing with the special case when the initial data $\alpha(r)$ and $\beta(r)$ and the solution $u(r)$ are functions only of $r$ and not of $\theta$. In this special case, the solution $u$ is given by the relatively simple formulas in (7.7.64) and (7.7.66). However, in case $\alpha$, $\beta$, and $u$ are functions both of $r$ and $\theta$, the formula for $u$ would be more complicated — it would look like equation (7.7.46), but with two extra double sums added in which $\cos c\sqrt{\lambda_{mn}}t$ is replaced by $\sin c\sqrt{\lambda_{mn}}t$. There would then be not just the two sets of constants $A_{mn}$ and $B_{mn}$ to find, but two additional sets of constants $C_{mn}$ and $D_{mn}$ as well.

**8.2. Heat flow with sources and nonhomogeneous boundary conditions.** The homogeneous heat equation is

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2};$$

the inhomogeneous heat equation is

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + Q(x, t).$$

Homogeneous Dirichlet boundary conditions are of the form $u(0, t) = 0$ and $u(L, t) = 0$; inhomogeneous Dirichlet boundary conditions are of the form $u(0, t) = A(t)$ and $u(L, t) = B(t)$. There are also homogeneous and inhomogeneous Neumann or mixed boundary condtions.

In class I explained this material slightly differently than the text does on pages 347 to 350. What I did in class is summarized in the paragraph titled “Related homogeneous boundary conditions” on page 351. Basically, the idea is that you can transform a problem with inhomogeneous boundary conditions and (possibly) inhomogeneous differential equation into another in which the differential equation is still inhomogeneous (or becomes inhomeogenous when it wasn’t before), but now the boundary conditions are homogeneous. You can just review this and skip straight to section 8.3.

**8.3. Eigenfunction expansion with homogeneous boundary conditions.** You should read through the entire section, including the example. I’ll also try to post a solution to one of the homework problems from Assignment 8 by the day before the exam.