SUPPLEMENTARY PROBLEMS

MATH 4163, INTRODUCTION TO PDE

1. Fourier series

1.1. Are the following pairs of functions orthogonal over the interval indicated?
a. 1 and x, $[-2,2]$
b. 1 and x, $[0,2]$
c. $\sin x$ and $\sin 2x$, $[0,\pi]$
d. $\sin x$ and $\cos x$, $[0,\pi]$
e. $\sinh x$ and $\cosh x$, $[-1,1]$
f. $P_2(x) = (1/2)(3x^2-1)$ and $P_3(x) = (1/2)(5x^3-3x)$, $[-1,1]$

1.2.
a. Prove: The system of functions $\{\sin \frac{\pi n x}{L}\}_{n=1}^{\infty}$ is orthogonal on interval $[0,L]$.
b. Normalize this system.

1.3.
a. Prove: The system of functions $\{\cos \frac{\pi n x}{L}\}_{n=0}^{\infty}$ is orthogonal on interval $[0,L]$.
b. Normalize this system.

1.4. Show that the functions $\psi_1(x) = 1$ and $\psi_2(x) = x$ are orthogonal on the interval $-1 < x < 1$, and determine constants A and B such that the function $\psi_3(x) = 1 + Ax + Bx^2$ is orthogonal to both ψ_1 and ψ_2 on that interval.

1.5. Find the Fourier cosine series of the following functions over the interval indicated. Sketch the graph of the series for at least 3 periods. Find the Fourier sine series of the following functions over the interval indicated. Sketch the graph of the series for at least 3 periods.
a. x, $[0,\pi]$
b. 1, $[0,2]$
c. x^2, $[0,1]$
d. e^{-x}, $[0,2]$

2. Superposition Principle

2.1. Suppose that the functions $u_n = u_n(x,y)$, $n = 1,2,\ldots$, are all solutions of Laplace's equation

$$u_{xx} + u_{yy} = 0.$$

Show that for any constants c_n, $n = 1,2,\ldots$, the linear combination

$$u = \sum_{n=1}^{N} c_n u_n$$

is also a solution.

2.2. Verify that each of the functions

$$u_0(x,y) = y, \quad u_n(x,y) = \sinh ny \cos nx \quad n = 1, 2, \ldots$$

satisfies Laplace's equation

$$u_{xx} + u_{yy} = 0, \quad 0 < x < \pi, \ 0 < y < 2$$

and the three boundary conditions

$$u_x(0,y) = u_x(\pi,y) = 0, \quad u(x,0) = 0.$$

Show that any linear combination

$$u(x,y) = A_0 y + \sum_{n=1}^{N} A_n \sinh ny \cos nx$$

satisfies the same differential equation and boundary conditions.

2.3. Verify that each of the functions

$$u_n(x,t) = e^{-\frac{(2n-1)^2\pi^2}{4}t} \sin \frac{(2n-1)\pi x}{2}, \quad n = 1, 2, \ldots$$

satisfies the heat equation

$$u_t = u_{xx}, \quad 0 < x < 1, \ t > 0$$

and the boundary conditions

$$u(0,t) = u_x(1,t) = 0.$$

Show that any linear combination

$$u(x,t) = \sum_{n=1}^{N} u_n(x,t)$$

satisfies the same differential equation and boundary conditions.

2.4. Verify that each of the functions

$$u_{mn}(x,y,z) = e^{-z\sqrt{m^2+n^2}} \cos my \sin nx, \quad m = 0, 1, 2, \ldots \quad n = 1, 2, \ldots$$

satisfies Laplace's equation

$$u_{xx} + u_{yy} + u_{zz} = 0$$

and the boundary conditions

$$u(0,y,z) = u(\pi,y,z) = 0, \quad u_y(x,0,z) = u_y(x,\pi,z) = 0.$$

Show that any linear combination

$$u(x,y,z) = \sum_{n=1}^{N} u_n(x,y,z)$$

satisfies the same differential equation and boundary conditions.

TABLE 1. Thermal diffusivity constants k

Material	k (cm^2/s)
Silver	1.70
Copper	1.15
Aluminum	0.85
Iron	0.15
Concrete	0.005

3. HEAT EQUATION

3.1. Suppose that a rod $40cm$ long with insulated lateral surface is heated to a uniform temperature of 100^oC, and that at time $t = 0$ its two ends are embedded in ice at 0^oC.
a. Find the temperature $u(x,t)$ of the rod.
b. In the case the rod is made of copper, show that after 5 min the temperature at the midpoint is about 15^oC.
c. In the case the rod is made of concrete, use the first term of the series to find the time required for its midpoint to cool to 15^oC.

3.2. A copper rod $50cm$ long with insulated lateral surface has initial temperature of $u(x,0) = 2x$, and at time $t = 0$ its two ends are insulated.
a. Find the temperature $u(x,t)$ of the rod.
b. What will its temperature be at $x = 20$ after 1 min?
c. How long will it take for the temperature at $x = 20$ to reach 45^oC?

3.3. Two slabs of iron, each $10cm$ thick, are such that one is at 100^oC and the other at 0^oC. They are placed face to face in perfect contact, and their outer faces are kept at 0^oC.
a. Find the temperature $u(x,t)$ in the slabs.
b. What will the temperature at the common face be at after 15 min?
c. If the slabs are made of concrete, what will the temperature at the common face be at after 1 h?

3.4. The temperature $u(x,t)$ in a bare slender wire the rate of heat loss through the lateral surface proportional to its temperature u satisfies

$$u_t = ku_{xx} - hu, \quad 0 < x < L,\ t > 0$$
$$u(0,t) = u(L,t) = 0$$
$$u(x,0) = f(x).$$

Find the temperature $u(x,t)$ inside the wire. Here $h > 0$.

3.5. The voltage $e(x,t)$ along a submarine cable 2000 kilometers long satisfies

$$e_{xx} = RCe_t, \quad 0 < x < 2000,\ t > 0$$
$$e(0,t) = e(2000,t) = 0$$
$$e(x,0) = \sin\frac{x}{100}.$$

Find the voltage $e(x,t)$.

3.6. The current $i(x,t)$ along a submarine cable 1000 kilometers long satisfies

$$i_{xx} = RCi_t, \quad 0 < x < 1000,\ t > 0$$
$$i_x(0,t) = i_x(1000,t) = 0$$
$$i(x,0) = \sin\frac{x}{100}.$$

Find the current $i(x,t)$.

4. Wave Equation

4.1. A vibrating string is fastened to air bearings situated on vertical rods at $x = 0$ and $x = 2$. Find the displacement $u(x,t)$ if the conditions are

$$u_{tt} = u_{xx}, \quad 0 < x < 2,\ t > 0$$
$$u_x(0,t) = u_x(2,t) = 0$$
$$u(x,0) = x,\ u_t(x,0) = 0$$

4.2. The pressure $p(x,t)$ in an organ pipe satisfies

$$p_{xx} = \frac{1}{c^2}p_{tt}$$

If the pipe is L meters long and open at both ends, find the pressure $p(x,t)$ if $p(x,0) = 0$ and $p_t(x,0) = 40$.

4.3. The length of a guitar string is $65cm$. If the string is plucked $15cm$ from the bridge (i.e. the end of the wire) by raising it $3mm$, find the displacement $u(x,t)$.

4.4. A string vibrating in air with resistance proportional to velocity satisfies

$$u_{tt} = a^2u_{xx} - 2hu_t, \quad 0 < x < L,\ t > 0$$
$$u(0,t) = u(L,t) = 0$$
$$u(x,0) = f(x),\ u_t(x,0) = 0$$

Assume

$$0 < h < \frac{\pi a}{L}$$

Find the displacement $u(x,t)$.

5. Laplace's Equation

6. Sturm-Liouville problems

6.1. Solve the following problem ($b > 1$)

$$r^2u_{rr}(r,\theta) + ru_r(r,\theta) + u_{\theta\theta}(r,\theta) = 0, \quad 1 < r < b,\ 0 < \theta < \pi$$
$$u(r,0) = 0,\ u(r,\pi) = r$$
$$u(1,\theta) = 0,\ u(b,\theta) = 0$$

6.2. Solve the following problem

$$u_t(x,t) = ku_{xx}(x,t), \quad 0 < x < \pi,\ t > 0$$
$$u(0,t) = 0,\ u_x(\pi,t) = -hu(\pi,t),\ u(x,0) = f(x)$$

6.3. Solve the following problem ($h > 0$)

$$u_t(x,t) = ku_{xx}(x,t), \quad 0 < x < 1,\ t > 0$$
$$u_x(0,t) = 0,\ u_x(1,t) = -hu(1,t),\ u(x,0) = f(x)$$

6.4. Solve the following problem ($h > 0$)

$$u_{xx} + u_{yy} = 0, \quad 0 < x < a,\ 0 < y < b$$
$$u_x(0,y) = 0,\ u_x(a,y) = -hu(a,y)$$
$$u(x,0) = 0,\ u(x,b) = f(x)$$

6.5. Find a bounded harmonic function in the semi-infinite strip $x > 0,\ 0 < y < 1$, satisfying

$$u(x,0) = 0,\ u_y(x,1) = -hu(x,1),\ u(0,y) = u_0$$

where $h > 0$.

6.6. Find a bounded harmonic function in the semi-infinite strip $0 < x < 1,\ y > 0$, satisfying

$$u_x(0,y) = 0,\ u_x(1,y) = -hu(1,y),\ u(x,0) = f(x)$$

where $h > 0$.

6.7. Solve the following problem

$$(t+1)u_t(x,t) = u_{xx}(x,t), \quad 0 < x < \pi,\ t > 0$$
$$u(0,t) = 0,\ u_x(\pi,t) = 0,\ u(x,0) = 1$$

6.8. Write the following differential equations in self-adjoint form

a. $y'' + 2y' + 3y = 0$
b. $xy'' + y' + (x - \lambda)y = 0$
c. $x^2y'' + xy' + 10y = 0$
d. $x^2y'' + xy' + (x^2 - \lambda^2)y = 0$ Bessel's equation
e. $(1 - x^2)y'' - 2xy' + n(n+1)y = 0$ Legendre's equation
f. $y'' + \lambda y = 0$
g. $y'' - xy = 0$ Airy's equation
h. $(1 - x^2)y'' - xy' + n^2y = 0$ Chebyshev's equation
i. $(1 - x^2)y'' - 3xy' + n(n+2)y = 0$ Chebyshev's equation
j. $xy'' + (1 - x)y' + ny = 0$ Laguerre's equation
k. $y'' - 2xy' + 2ny = 0$ Hermite's equation

7. Bessel Functions

7.1. Establish the differentiation formula

$$\int_0^x s^n J_0(s)\, ds = x^n J_1(x) + (n-1)x^{n-1}J_0(x) - (n-1)^2 \int_0^x s^{n-2}J_0(s)\, ds$$

$n = 2, 3, \ldots.$

7.2. Find

$$\int_0^x s^3 J_0(s)\, ds$$

7.3. Prove

$$\int_0^x s^5 J_0(s)\, ds = x(x^2 - 8)[4xJ_0(x) + (x^2 - 8)J_1(x)]$$

7.4. Find the Fourier-Bessel series on interval $(0, c)$

$$100 = \sum_{j=1}^{\infty} A_j J_0(\alpha_j x), \quad J_0(\alpha_j c) = 0$$

7.5. Find the Fourier-Bessel series on interval $(0, 5)$

$$x = \sum_{j=1}^{\infty} A_j J_1(\alpha_j x), \quad J_1(5\alpha_j) = 0$$

7.6. Find the Fourier-Bessel series on interval $(0, c)$

$$10 = \sum_{j=1}^{\infty} A_j J_2(\alpha_j x), \quad J_2(\alpha_j c) = 0$$

7.7. Find the Fourier-Bessel series on interval $(0, c)$.

$$x^2 = \sum_{j=1}^{\infty} A_j J_0(\alpha_j x), \quad J_0(\alpha_j c) = 0$$

7.8. Suppose that a circular membrane $u(r,t)$, $0 < r < c$ has initial position $u(r,0) = f(r)$ and the initial velocity $u_t(r,0) = 0$. Find the deflection $u(r,t)$, assuming $u(c,t) = 0$, $t > 0$.

7.9. Suppose that a circular membrane $u(r,t)$, $0 < r < c$ has initial position $u(r,0) = 0$ and the initial velocity $u_t(r,0) = v_0$. Find the deflection $u(r,t)$, assuming $u(c,t) = 0$, $t > 0$.

7.10.
a. Suppose that a circular membrane $u(r,t)$, $0 < r < c$ has initial position $u(r,0) = 0$ and the initial velocity

$$u_t(r,0) = \begin{cases} \frac{P_0}{\pi\epsilon^2} & \text{if } 0 \le r < \epsilon, \\ 0 & \text{if } \epsilon < r \le c. \end{cases}$$

Find the deflection $u(r,t)$, assuming $u(c,t) = 0$, $t > 0$.
b. Use the fact that $[J_1(x)]/x \to 1/2$ as $x \to 0$ to find the limiting value of the result in part (a) as $\epsilon \to 0$. This describes the motion of a drumhead resulting from an initial momentum impulse P_0 at its center.

7.11. A round hamburger 12 cm in diameter is inserted in a roll (a perfect insulator). Initially the hamburger is at 180^oC. Since it is too hot to eat, the hamburger is taken outside, where the temperature is 0^oC. Find the temperature of the hamburger $u(r,t)$. How long will it take for the hamburger to cool down to below 50^oC? Use only the first term in the series and $k = 0.02cm^2/s$.

7.12. Find the steady state temperature $u(r,z)$ in the solid cylinder formed by the three surfaces $r = 1$, $z = 0$, $z = 1$, if $u = 0$ on the side, the bottom is insulated, and $u = u_0$ at the top.

7.13. Find the steady state temperature $u(r,z)$ in the semi-infinite cylinder $r \le 1$, $z \ge 0$, if $u = 1$ on the base, and the heat transfer on the surface $r = 1$ satisfies $u_r(1,z) = -hu(1,z)$.

7.14. Find a harmonic function $u(r, z)$ in the interior of the cylinder formed by the three surfaces $r = c$, $z = 0$, $z = b$, if $u = 0$ on the side, $u = 0$ on the bottom, and $u = f(r)$ at the top.

8. Nonhomogeneous Problems

8.1. The initial temperature of a slab $0 \leq x \leq \pi$ is zero. The face $x = 0$ is kept at temperature 0, and heat is supplied through the face $x = \pi$ at a constant rate $A > 0$ per unit area, so that $Ku_x(\pi, t) = A$. Find the temperature $u(x, t)$ in the slab.

8.2. Solve the following problem $(b > 0)$

$$u_t = u_{xx} - bu, \quad 0 < x < \pi,\ t > 0$$
$$u(0,t) = 0,\ u(\pi, t) = 1,\ u(x, 0) = 0$$

8.3. Solve the following problem

$$u_t = u_{xx} + xp(t), \quad 0 < x < 1,\ t > 0$$
$$u(0,t) = 0,\ u(1,t) = 1,\ u(x,0) = 0$$

8.4. Let $u(x, t)$ denote temperature in a slab $0 \leq x \leq 1$ that is initially at temperature zero, and whose faces are at temperatures

$$u(0,t) = 0,\ u(1,t) = F(t)$$

Find $u(x, t)$, if $F(t)$ and $F'(t)$ are continuous with $F(0) = 0$.

8.5. Heat is generated in a cylinder at a constant rate $q_0 > 0$ per unit of volume. Find the temperature $u(r, t)$ in the cylinder:

$$u_t = k(u_{rr} + \frac{1}{r}u_r) + q_0, \quad 0 < r < c,\ t > 0$$
$$u(c,t) = 0,\ u(r,0) = 0$$

8.6. Solve the previous problem if the heat generation rate is time dependent (replace q_0 by $q(t)$).

9. D'Alembert solution

9.1. Show that

$$u(x,t) = \frac{f(x+at) + f(x-at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at} g(s)\, ds$$

satisfies the wave equation

$$u_{tt} = a^2 u_{xx}, \quad -\infty < x < \infty, \quad t > 0,$$
$$u(x,0) = f(x), \quad u_t(x,0) = g(x).$$

9.2. Let $g(x) = 0$ in Problem (9.1). Derive the solution $u(x, t)$ for (9.1) using the Fourier Transform.

9.3. An infinite string is initially at rest along the x axis. The string is lifted over the interval $(0, 2)$ to form an equilateral triangle, and then released. Sketch the string position for various times. Use $a = 1$.