Math 4163 – Solutions to Assignment 9

$\boxed{7.7.3(c)}$ Solve $u_{tt} = c^2 \Delta u$ on $0 < r < a$, $0 < \theta < \frac{\pi}{2}$, with boundary conditions

$$u(r,0,t) = 0, \quad u(r,\tfrac{\pi}{2},t) = 0, \quad \frac{\partial u}{\partial r}(a,\theta,t) = 0$$

and initial conditions

$$u(r,\theta,0) = 0 \quad \text{and} \quad \frac{\partial u}{\partial t}(r,\theta,0) = \beta(r,\theta)$$

You may assume that product solutions

$$u(r,\theta,t) = \varphi(r,\theta)\,h(t) = f(r)\,g(\theta)\,h(t)$$

satisfy

$$* \quad \Delta\varphi + \lambda\varphi = 0 \qquad (\lambda > 0)$$

$$* \quad \frac{d^2h}{dt^2} = -c^2\lambda h$$

$$* \quad \frac{d^2g}{d\theta^2} = -\mu g$$

$$* \quad r\frac{d}{dr}\left(r\frac{df}{dr}\right) + (\lambda r^2 - \mu) f = 0$$

Solution: The boundary conditions for $g(\theta)$ are $g(0) = 0$ and $g(\frac{\pi}{2}) = 0$, so the eigenvalues μ for the Sturm-Liouville equation for g are

for $(m = 1, 2, 3, \dots)$ $\mu = \left(\frac{m\pi}{\pi/2}\right)^2 = (2m)^2$, and the eigenfunctions

are $g(\theta) = \sin(\sqrt{\mu}\,\theta) = \sin(2m\theta)$.

Putting $z = \sqrt{\lambda}\,r$ and $\mu = (2m)^2$ in the equation for f, we get, for $\tilde{f}(z) = f(r)$,

$$z^2\frac{d^2\tilde{f}}{dz^2} + z\frac{d\tilde{f}}{dz} + (z^2 - (2m)^2)\tilde{f} = 0.$$

cont'd →

This is Bessel's equation of order $2m$. The general solution is $A J_{2m}(z) + B Y_{2m}(z) = \tilde{f}(z)$,
so $f(r) = \tilde{f}(\sqrt{\lambda} r) = A J_{2m}(\sqrt{\lambda} r) + B Y_{2m}(\sqrt{\lambda} r)$.

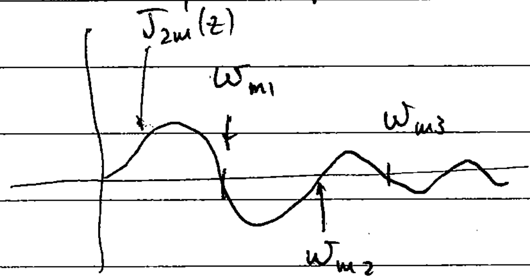
Since $\lim_{r \to 0} f(r)$ should be finite we must have

$B = 0$, and so $f(r) = A J_{2m}(\sqrt{\lambda} r)$.

Since $\frac{\partial u}{\partial r}(a, \theta, t) = 0$, we get $f'(a) = 0$,

so $A\sqrt{\lambda}\, J_{2m}'(\sqrt{\lambda} a) = 0$, so

$z = \sqrt{\lambda} a$ must be a point where $J_{2m}'(z) = 0$



Let w_{mn}, $n = 1, 2, 3, \dots$ be the zeroes of $J_{2m}'(z)$. Then $\sqrt{\lambda}\, a = w_{mn}$, so $\lambda = \left(\frac{w_{mn}}{a}\right)^2$,

which we call λ_{mn}. Then $\lambda = \lambda_{mn}$ and

$$f(r) = A J_{2m}(\sqrt{\lambda_{mn}}\, r).$$

Also, from the equation for $h(t)$ with $\lambda = \lambda_{mn}$, we get $h(t) = A\cos(c\sqrt{\lambda_{mn}}\, t) + B\sin(c\sqrt{\lambda_{mn}}\, t)$

so product solutions are

$$u(r,\theta,t) = J_{2m}(\sqrt{\lambda_{mn}}\, r) \cdot \sin(2m\theta) \left[A\cos(c\sqrt{\lambda_{mn}}\, t) + B\sin(c\sqrt{\lambda_{mn}}\, t)\right]$$

To satisfy the initial conditions, take

$$u(r,\theta,t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} J_{2m}(\sqrt{\lambda_{mn}}\, r) \sin(2m\theta) \left[A_{mn}\cos(c\sqrt{\lambda_{mn}}\, t) + B_{mn}\sin(c\sqrt{\lambda_{mn}}\, t)\right]$$

Since $u(r,\theta,0) = 0$, we get $A_{mn} = 0$ for all m, n.

So

$$\boxed{u(r,\theta,t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} J_{2m}(\sqrt{\lambda_{mn}}\, r) \sin(2m\theta)\, B_{mn} \sin(c\sqrt{\lambda_{mn}}\, t)}$$

and

$$\frac{\partial u}{\partial t}(r,\theta,t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} J_{2m}(\sqrt{\lambda_{mn}}\, r) \sin(2m\theta)\, c\sqrt{\lambda_{mn}}\, B_{mn} \cos(c\sqrt{\lambda_{mn}}\, t)$$

Putting $t = 0$ gives

$$\beta(r,\theta) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} J_{2m}(\sqrt{\lambda_{mn}}\, r) \sin(2m\theta)\, c\sqrt{\lambda_{mn}}\, B_{mn} .$$

Multiplying by $\varphi_{\bar m \bar n} = J_{2\bar m}(\sqrt{\lambda_{\bar m \bar n}}\, r) \sin(2\bar m \theta)$ and integrating over the quarter-disc, using orthogonality, we find

$$B_{\bar m \bar n} = \frac{1}{c\sqrt{\lambda_{\bar m \bar n}}} \frac{\int_0^{\frac{\pi}{2}} \int_0^a J_{2\bar m}(\sqrt{\lambda_{\bar m \bar n}}\, r) \sin(2\bar m\theta)\, \beta(r,\theta)\, r\, dr\, d\theta}{\int_0^{\pi/2} \int_0^a J_{2\bar m}(\sqrt{\lambda_{\bar m \bar n}}\, r)^2 \sin^2(2\bar m\theta)\, r\, dr\, d\theta}$$

$\boxed{7.7.9(a)}$ Putting $u(r,\theta,t) = f(r)\,g(\theta)\,h(t)$, we get, after separating variables,

$$* \quad \frac{dh}{dt} = -\lambda k h$$

$$* \quad \frac{d^2 g}{d\theta^2} = -\mu g$$

$$* \quad r\frac{d}{dr}\left(r\frac{df}{dr}\right) + (\lambda r^2 - \mu) f = 0$$

The boundary conditions for g are $g(0)=0$ and $g(\pi)=0$, so $\mu = -m^2$ $(m = 1,2,3,\dots)$ and $g(\theta) = \sin(m\theta)$.

As for the wave equation, putting $z = \sqrt{\lambda}\, r$ in the equation for f gives

$$z^2 \frac{d\tilde{f}}{dz^2} + z\frac{d\tilde{f}}{dz} + (z^2 - m^2)\tilde{f} = 0$$

where $\tilde{f}(\sqrt{\lambda}r) = f(r)$, and since $\lim_{r\to 0} f(r) = 0$ we must have $\tilde{f}(z) = A\,J_m(z)$ and $f(r) = A\,J_m(\sqrt{\lambda}r)$.

The boundary condition $\frac{\partial u}{\partial r}(a,\theta,t) = 0$ gives $f'(a) = 0$, so

$A\sqrt{\lambda}\,J_m'(\sqrt{\lambda}a) = 0$, so $\sqrt{\lambda}a = z$ is a solution of $J_m'(z) = 0$. Let y_{mn} $(n = 1,2,\dots)$ denote the solutions of $J_m'(z) = 0$; Then

$\lambda = \lambda_{mn} = \left(\frac{y_{mn}}{a}\right)^2$, and $f(r) = A\,J_m(\sqrt{\lambda_{mn}}\,r)$

Also from the equation for $h(t)$ we get

$$h(t) = C e^{-k\lambda_{mn}t}.$$

So $u_{mn}(r,\theta,t) = J_m(\sqrt{\lambda_{mn}}\,r)\sin(m\theta)\,e^{-k\lambda_{mn}t}$ are the separated solutions.

To satisfy the initial condition, put

$$\boxed{u(r,\theta,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{mn}\, J_m(\sqrt{\lambda_{mn}}\, r)\sin m\theta\, e^{-k\lambda_{mn}t}}$$

Putting $t=0$ gives

$$f(r,\theta) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{mn}\, J_m(\sqrt{\lambda_{mn}}\, r)\sin m\theta\,,$$

and using orthogonality, we find

$$\boxed{C_{mn} = \frac{\int_0^{\pi}\int_0^{a} f(r,\theta)\, J_m(\sqrt{\lambda_{mn}}\, r)\sin(m\theta)\, r\,dr\,d\theta}{\int_0^{\pi}\int_0^{a} J_m(\sqrt{\lambda_{mn}}\, r)^2 \sin^2(m\theta)\, r\,dr\,d\theta}}$$