EXAM 1
Math 4163
2-15-13

Name Key

Instructions *Work all of the following problems in the space provided. If there is not enough room, you may write on the back sides of the pages. Give thorough explanations to receive full credit.*

1. *(20 points)* Consider the heat equation

$$\frac{\partial u}{\partial t} = 10\,\frac{\partial^2 u}{\partial x^2},$$

for $0 < x < 2\pi$, $t > 0$, subject to the boundary conditions

$$u(0,t) = 0 \quad \text{and} \quad u(2\pi, t) = 0$$

for all $t > 0$. Find $u(x,t)$ if the initial data is $u(x,0) = 5\sin(3x/2) - 7\sin(2x)$ for $0 \le x \le 2\pi$.

We know from class that separated solutions of $u_t = k u_{xx}$ with $u(0,t)=0$, $u(L,t)=0$ are $u(x,t) = B \sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$ 5 $(n=1,2,3,\dots)$

Here $k=10$ and $L=2\pi$, so separated solutions are 5 $u(x,t) = B\sin\left(\frac{nx}{2}\right)e^{-10\left(\frac{n}{2}\right)^2 t}$.

The initial data $u(x,0) = 5\sin\left(\frac{3x}{2}\right) + (-7)\sin\left(\frac{4x}{2}\right)$, so we should take $u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{nx}{2}\right) e^{-10\left(\frac{n}{2}\right)^2 t}$ with 5 $B_3 = 5$, $B_4 = (-7)$, and all the other B_n's equal to zero. So 5 $u(x,t) = 5\sin\left(\frac{3x}{2}\right)e^{-10\left(\frac{3}{2}\right)^2 t} + (-7)\sin\left(\frac{4x}{2}\right)e^{-10\left(\frac{4}{2}\right)^2 t}$

or $\boxed{u(x,t) = 5\sin\left(\frac{3x}{2}\right)e^{-\frac{45}{2}t} - 7\sin(2x)e^{-40t}}$

2. *(20 points)* Suppose

$$u(x,y) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)\cosh\left(\frac{n\pi y}{L}\right).$$

a) Give a formula for $\frac{\partial u}{\partial y}(x,y)$ as the sum of a series.

$$\frac{\partial u}{\partial x}(x,y) = \sum_{n=1}^{\infty} \frac{\partial}{\partial y}\left(B_n \sin\left(\frac{n\pi x}{L}\right)\cosh\left(\frac{n\pi y}{L}\right)\right) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)\frac{n\pi}{L}\sinh\left(\frac{n\pi y}{L}\right)$$

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b) If $\frac{\partial u}{\partial y}(x,H) = g(x)$ for $0 \le x \le L$, where $g(x)$ is a given function, find formulas for the coefficients B_n.

Putting $y=H$ in the formula from a) gives 4

$$g(x) = \frac{\partial u}{\partial x}(x,H) = \sum_{n=1}^{\infty} B_n \left(\frac{n\pi}{L}\right)\sinh\left(\frac{n\pi H}{L}\right)\sin\left(\frac{n\pi x}{L}\right).$$

Multiplying both sides by $\sin\left(\frac{m\pi x}{L}\right)$ and integrating from $x=0$ to $x=L$ gives

$$\int_0^L g(x)\sin\left(\frac{m\pi x}{L}\right)dx \overset{4}{=} B_m\left(\frac{m\pi}{L}\right)\sinh\left(\frac{m\pi H}{L}\right)\cdot\frac{L}{2} \quad 4$$

(after using orthogonality and $\int_0^L \sin^2\left(\frac{m\pi x}{L}\right)dx = \frac{L}{2}$). So

$$B_n = \frac{2}{L}\cdot\frac{L}{m\pi}\cdot\frac{1}{\sinh\left(\frac{m\pi H}{L}\right)} \overset{4}{\int_0^L} g(x)\sin\left(\frac{m\pi x}{L}\right)dx = \boxed{\frac{2}{m\pi\sinh\left(\frac{m\pi H}{L}\right)}\int_0^L g(x)\sin\left(\frac{m\pi x}{L}\right)dx}$$

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3. *(30 points)* Consider the heat equation
$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2},$$
for $0 < x < L$, $t > 0$, subject to the boundary conditions
$$\frac{\partial u}{\partial x}(0,t) = 0 \quad \text{and} \quad u(L,t) = 0.$$
You are given (you do not need to derive this yourself!) that the eigenfunctions for these boundary conditions are
$$\cos\left(\frac{n\pi x}{2L}\right) \quad \text{for } n = 1, 3, 5, 7, \ldots.$$

a) Find, in series form, the solution $u(x,t)$ satisfying the initial condition

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$$u(x,0) = 100 \quad \text{for } 0 \le x \le L.$$

Separated solutions $u(x,t) = \varphi(x)h(t)$ of the heat equation satisfy $\varphi''(x) = -\lambda\varphi(x)$ and $h'(t) = -k\lambda h(t)$. Here the boundary conditions are $\varphi'(0) = 0$ and $\varphi(L) = 0$, and we are given that eigenfunctions are $\varphi(x) = \cos\left(\frac{n\pi}{2L}x\right)$ for n odd, so $\lambda = +\left(\frac{n\pi}{2L}\right)^2$ for n odd. Then

$h(t) = De^{-k\lambda t} = De^{-k\left(\frac{n\pi}{2L}\right)^2 t}$ 5 for n odd. Thus we take

$$\boxed{u(x,t) = \sum_{n=1,3,5,\ldots} B_n \cos\left(\frac{n\pi}{2L}x\right) e^{-k\left(\frac{n\pi}{2L}\right)^2 t}}$$ 5

Putting $t=0$ gives $100 = \sum_{n \text{ odd}} B_n \cos\left(\frac{n\pi}{2L}x\right)$. 3 From orthogonality,

$B_n = \frac{2}{L}\int_0^L 100\cos\left(\frac{n\pi}{2L}s\right)ds$ 3 $= \frac{200}{L}\int_0^L \cos\left(\frac{n\pi}{2L}s\right)ds$

$= \frac{200}{L}\frac{2L}{n\pi}\left[\sin\left(\frac{n\pi}{2L}s\right)\right]_{s=0}^{s=L}$ 3 $= \frac{400}{n\pi}\left[\sin\left(\frac{n\pi}{2}\right) - \sin(0)\right]$ 3 $= \boxed{\frac{400}{n\pi}\sin\left(\frac{n\pi}{2}\right)}$

$(n = 1, 3, 5, 7, \ldots)$

b) Write out explicitly the first three non-zero terms of the series for $u(x,t)$.

[8] Since $\sin\left(\frac{n\pi}{2}\right) = \begin{cases} 1 & \text{for } n = 3, 7, 11, \ldots \\ -1 & \text{for } n = 5, 9, 13, \ldots \end{cases}$

Then $(B_1, B_3, B_5, \ldots) = \left(\frac{400}{\pi}\cdot 1,\ \frac{400}{3\pi}(-1),\ \frac{400}{5\pi}(1),\ \frac{400}{7\pi}(-1), \ldots\right)$

So
$$\boxed{u(x,t) = \frac{400}{\pi}\cos\left(\frac{\pi x}{2L}\right)e^{-k\left(\frac{\pi}{2L}\right)^2 t} - \frac{400}{3\pi}\cos\left(\frac{3\pi x}{2L}\right)e^{-k\left(\frac{3\pi}{2L}\right)^2 t} + \frac{400}{5\pi}\cos\left(\frac{5\pi x}{2L}\right)e^{-k\left(\frac{5\pi}{2L}\right)^2 t} - \ldots}$$

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4. *(30 points)* Consider Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

for $0 < x < L$ and $0 < y < H$.

a) Find all solutions of the form $u(x,y) = \phi(x)h(y)$ of the equation, subject to the boundary conditions

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$$\frac{\partial u}{\partial x}(0,y) = 0, \quad \frac{\partial u}{\partial y}(x,0) = 0, \quad \text{and} \quad \frac{\partial u}{\partial y}(x,H) = 0.$$

If $u(x,y) = \varphi(x)h(y)$ solves Laplace's equation, then

$$h''(y) = -\lambda h(y) \text{ and } \varphi''(x) = \lambda \varphi(x).$$

We put the minus sign in the equation for $h(y)$ because $h(y)$ satisfies two boundary conditions $h'(0)=0$ 2 and $h'(H)=0$. 2 From class we know that the two-point boundary-value problem for $h(y)$ has eigenfunctions $h(y) \equiv 1$ (for $\lambda = 0$) and $h(y) = \cos\left(\frac{n\pi y}{H}\right)$ (for $\lambda = \left(\frac{n\pi}{H}\right)^2$, $n = 1, 2, 3, \ldots$. 3

The boundary condition $\frac{\partial u}{\partial x}(0,y) = 0$ gives $\varphi'(0) = 0$. 2

– For $\lambda = 0$, we have $\varphi''(x) = 0 \Rightarrow \varphi(x) = A + Bx$. So $\varphi'(x) = B$, and $\varphi'(0) = 0 \Rightarrow B = 0$. So $\varphi(x) = A$. So $\boxed{u(x,y) = A \cdot 1}$ 3

– For $\lambda = \left(\frac{n\pi}{H}\right)^2$, $\varphi''(x) = \left(\frac{n\pi}{H}\right)^2 \varphi(x) \Rightarrow \varphi(x) = A\cosh\left(\frac{n\pi x}{H}\right) + B\sinh\left(\frac{n\pi x}{H}\right)$

So $\varphi'(x) = A\left(\frac{n\pi}{H}\right)\sinh\left(\frac{n\pi x}{H}\right) + B\left(\frac{n\pi}{H}\right)\cosh\left(\frac{n\pi x}{H}\right)$, and $\varphi'(0) = 0 \Rightarrow B = 0$.

So $\varphi(x) = A\cosh\left(\frac{n\pi x}{H}\right)$. So $\boxed{u(x,y) = A\cosh\left(\frac{n\pi x}{H}\right)\cos\left(\frac{n\pi y}{H}\right) \quad (n = 1, 2, 3, \ldots)}$ 3

b) Find a linear combination of the solutions from part *a)* that satisfies the condition $u(L,y) = f(y)$ for $0 \le y \le H$, where $f(y)$ is a given function. Give formulas for the coefficients.

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We take

$$\boxed{u(x,y) = A_0 + \sum_{n=1}^{\infty} A_n \cosh\left(\frac{n\pi x}{H}\right)\cos\left(\frac{n\pi y}{H}\right)}$$

Putting $x = L$ gives

$$f(y) = A_0 + \sum_{n=1}^{\infty} A_n \cosh\left(\frac{n\pi L}{H}\right)\cos\left(\frac{n\pi y}{H}\right)$$

Multiplying by 1 and integrating in y from 0 to H, using orthogonality, gives

$$\int_0^H f(y)\,dy = \int_0^H A_0\,dy = A_0 H \Rightarrow \boxed{A_0 = \frac{1}{H}\int_0^H f(y)\,dy}$$

Multiplying by $\cos\left(\frac{m\pi y}{H}\right)$ and integrating gives

$$\int_0^H f(y)\cos\left(\frac{m\pi y}{H}\right)dy = A_m \cosh\left(\frac{m\pi L}{H}\right)\cdot\frac{H}{2} \Rightarrow A_m = \frac{2}{H\cosh\left(\frac{m\pi L}{H}\right)}\int_0^H f(y)\cos\left(\frac{m\pi y}{H}\right)dy$$