ANSWERS TO EXAM 2

(2)

1. If $u(r,\theta) = \varphi(\theta)G(r)$, then $\frac{1}{r}\left(\varphi(\theta)G'(r) + r\varphi(\theta)G''(r)\right) + \frac{1}{r^2}\varphi''(\theta)G(r) = 0$,

so multiplying by r^2 and dividing by $\varphi(\theta)G(r)$ gives $\frac{rG'(r)}{G(r)} + \frac{r^2G''(r)}{G(r)} + \frac{\varphi''(\theta)}{\varphi(\theta)} = 0$ (2)

Separating variables gives $-\frac{[rG'(r) + r^2G''(r)]}{G(r)} = \frac{\varphi''(\theta)}{\varphi(\theta)} = -\lambda$, so $\varphi''(\theta) = -\lambda\varphi(\theta)$

and $r^2G''(r) + rG'(r) - \lambda G(r) = 0$. (2) The boundary conditions on u give

$\varphi(0) = 0$ and $\varphi(\pi) = 0$, (2) so the eigenvalues for the Sturm-Liouville problem

for φ are $\lambda = \left(\frac{n\pi}{\pi}\right)^2 = n^2$, (3) and the eigenfunctions are $\varphi(\theta) = \sin(n\theta)$. (2)

We know from class that the general solution of the equation

for $G(r)$ is $G(r) = Ar^n + Br^{-n}$, but since $u(r,\theta) \not\to \infty$ as $r \to 0$, we must

have $B = 0$. So $G(r) = Ar^n$, (2) and $u_n(r,\theta) = Ar^n\sin(n\theta)$. (2)

Take $\boxed{u(r,\theta) = \sum_{n=1}^{\infty} A_n r^n \sin(n\theta)}$. (2) Then

$$\frac{\partial u}{\partial r}(r,\theta) = \sum_{n=1}^{\infty} A_n n r^{n-1}\sin(n\theta), \text{ so } (2)$$

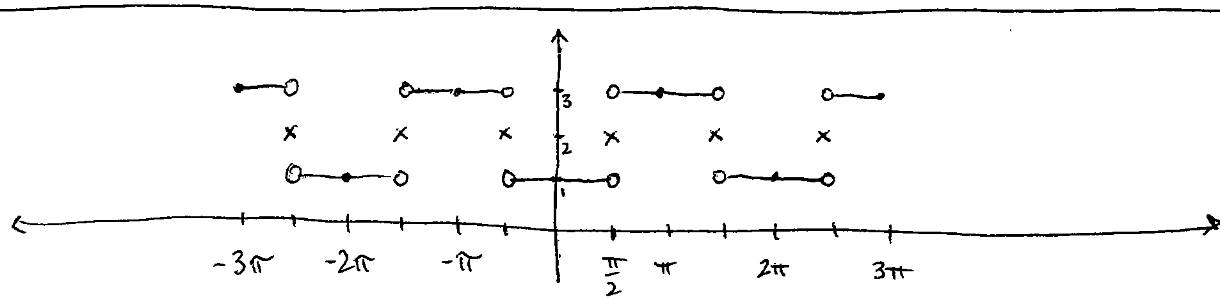
$$f(\theta) = \sum_{n=1}^{\infty} A_n n 2^{n-1}\sin(n\theta) \quad (2)$$

(2)

So $A_n \cdot n2^{n-1} = \frac{2}{\pi}\int_0^{\pi} f(\theta)\sin(n\theta)\,d\theta$, or $\boxed{A_n = \frac{2}{\pi n 2^{n-1}}\int_0^{\pi} f(\theta)\sin(n\theta)\,d\theta}$

---

2. a)



b)

$$A_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx = \frac{1}{\pi}\left[\int_0^{\pi/2} 1\,dx + \int_{\pi/2}^{\pi} 3\,dx\right] = \frac{1}{\pi}\left[\frac{\pi}{2} + 3\left(\pi - \frac{\pi}{2}\right)\right]$$

$$= \frac{1}{\pi}[2\pi] = \boxed{2}.$$

$$A_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos\left(\frac{n\pi x}{\pi}\right)dx = \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}}\cos(nx)\,dx + \int_{\frac{\pi}{2}}^{\pi} 3\cos(nx)\,dx\right] =$$

$$= \frac{2}{\pi}\left\{\frac{\sin(nx)}{n}\Big]_0^{\frac{\pi}{2}} + \frac{3\sin(nx)}{n}\Big]_{\frac{\pi}{2}}^{\pi}\right\} = \frac{2}{\pi}\left\{\frac{\sin(\frac{n\pi}{2})}{n} + \frac{3\sin(n\pi)}{n} - \frac{3\sin(\frac{n\pi}{2})}{n}\right\} = \boxed{\frac{-4}{n\pi}\sin\left(\frac{n\pi}{2}\right)}$$

(3) There was an error in this question. The equation should have been

$$\frac{1}{r}\left(\frac{\partial u}{\partial r} + r\frac{\partial^2 u}{\partial r^2}\right) + \frac{1}{r^2}\left(\frac{\partial^2 u}{\partial \theta^2} + 3\frac{\partial u}{\partial \theta}\right) = 0.$$

Then $u(r,\theta) = \varphi(\theta)G(r)$ would give $\frac{1}{r}\left(\varphi(\theta)G'(r) + r\varphi(\theta)G''(r)\right) + \frac{1}{r^2}\left(\varphi''(\theta)G(r) + 3\varphi'(\theta)G(r)\right) = 0$,

and dividing by $\varphi(\theta)G(r)$ and multiplying by r^2 gives

$$\frac{rG'(r)}{G(r)} + \frac{r^2G''(r)}{G(r)} + \frac{\varphi''(\theta)}{\varphi(\theta)} + \frac{3\varphi'(\theta)}{\varphi(\theta)} = 0, \text{ so}$$

$$\frac{rG'(r) + r^2G''(r)}{G(r)} = \frac{-[\varphi''(\theta) + 3\varphi'(\theta)]}{\varphi(\theta)} = \lambda = \text{constant, and}$$

$$\boxed{rG'(r) + r^2G''(r) = \lambda G(r)} \text{ and } \boxed{\varphi''(\theta) + 3\varphi'(\theta) = -\lambda\varphi(\theta)}.$$

For the equation given on the test, $\frac{1}{r}\left(\frac{\partial u}{\partial r} + r\frac{\partial^2 u}{\partial r^2}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + 3\frac{\partial u}{\partial \theta} = 0$, the variables can not be separated. Students who put $u(r,\theta) = \varphi(\theta)G(r)$ to get $\frac{1}{r}\left(\varphi(\theta)G'(r) + r\varphi(\theta)G''(r)\right) + \frac{1}{r^2}\varphi''(\theta)G(r) + 3\varphi'(\theta)G(r)$ and then made some attempt to separate the variables got the full 10 points for the problem. Students who commented that the equation did not seem to be separable got 2 points extra credit.

---

(4) a) Separating variables gives $\varphi''(x) = -\lambda\varphi(x)$ (2), with boundary conditions

[19] $\varphi(-L) = \varphi(L)$ and $\varphi'(-L) = \varphi'(L)$ (2), and $h''(t) = -c^2\lambda h(t)$ (2). From class we know the boundary-value problem for φ has eigenvalues $\lambda = \left(\frac{n\pi}{L}\right)^2$ (2) for $n = 0,1,2,3,\dots$; with eigenfunctions $\varphi_0(x) = A$ (1) for $n=0$ and

$\varphi_n(x) = \left(A\cos\frac{n\pi x}{L} + B\sin\frac{n\pi x}{L}\right)$ for $n = 1,2,3,\dots$ (2)

When $\lambda = 0$, $h''(t) = 0$, so $h(t) = C + Dt$ (2)

and $u(x,t) = \varphi(x)h(t) = A(C + Dt)$, or just $\boxed{u(x,t) = C + Dt}$ (2)

When $\lambda = \left(\frac{n\pi}{L}\right)^2$ $(n = 1,2,3,\dots)$, $h''(t) = -c^2\left(\frac{n^2\pi^2}{L^2}\right)h(t)$, so

$h(t) = C\cos\left(\frac{n\pi ct}{L}\right) + D\sin\left(\frac{n\pi ct}{L}\right)$ (2), and (2)

$$\boxed{u(x,t) = \left(A\cos\left(\frac{n\pi x}{L}\right) + B\sin\left(\frac{n\pi x}{L}\right)\right)\left(C\cos\left(\frac{n\pi ct}{L}\right) + D\sin\left(\frac{n\pi ct}{L}\right)\right)}$$

[6] b) Take $u(x,t) = C + Dt$. Then $u(x,0) = 1 \Rightarrow C + D\cdot 0 = 1$ (2) $\Rightarrow C = 1 \Rightarrow u(x,t) = 1 + Dt$

and $\frac{\partial u}{\partial t} = D$, so $\frac{\partial u}{\partial t}(x,0) = 3 \Rightarrow D = 3$ (2). So $\boxed{u(x,t) = 1 + 3t}$ (2)

[Note on problem (4): Several students tried to do more than the problem asked for, and give all solutions of $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ $(-L<x<L, t>0)$ with $\begin{cases} u(-L,t) = u(L,t) \text{ and} \\ \frac{\partial u}{\partial x}(-L,t) = \frac{\partial u}{\partial x}(L,t). \end{cases}$

The problem only asked for the solutions of the form $u(x,t) = \varphi(x)\, h(t)$ - The "separated" solutions.

— In case you are interested in finding all solutions, you have to be a little careful.

The answer is not, as you might expect,

$$u(x,t) = C_0 + D_0 t + \sum_{n=1}^{\infty} \left(A_n \cos\left(\frac{n\pi x}{L}\right) + B_n \sin\left(\frac{n\pi x}{L}\right)\right)\left(C_n \cos\left(\frac{n\pi c t}{L}\right) + D_n \sin\left(\frac{n\pi c t}{L}\right)\right)$$

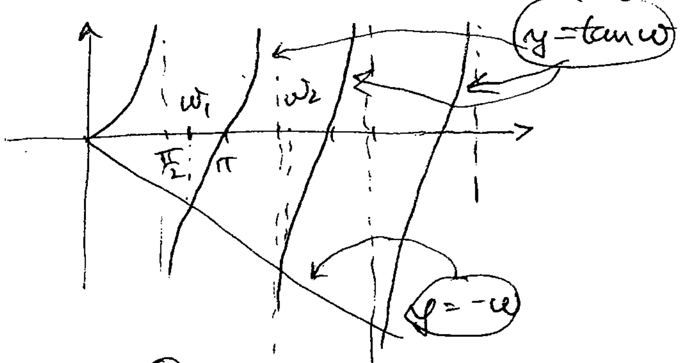
In fact, if you try to use the conditions $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$ to find the constants in this expression, you'll find in general that it's not possible. Instead, a formula which gives all the possible solutions is:

$$u(x,t) = C_0 + D_0 t + \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{n\pi x}{L}\right) + B_n \sin\left(\frac{n\pi x}{L}\right)\right] \cdot \cos\left(\frac{n\pi c t}{L}\right) + \sum_{n=1}^{\infty} \left[C_n \cos\left(\frac{n\pi x}{L}\right) + D_n \sin\left(\frac{n\pi x}{L}\right)\right] \cdot \sin\left(\frac{n\pi c t}{L}\right).]$$

(5) Separating variables, putting $u(x,t) = \varphi(x)\,G(t)$, gives $\varphi''(x) = -\lambda\,\varphi(x)$ and $G'(t) = -k\lambda G(t)$ (2). The boundary conditions for $\varphi(x)$ are $\varphi(0) = 0$ and $\varphi'(1) + \varphi(1) = 0$. (2)

We are given that the eigenvalues are positive, so $\lambda > 0$ and the general solution of the ODE for $\varphi(x)$ is $\varphi(x) = A\cos(\sqrt{\lambda}\,x) + B\sin(\sqrt{\lambda}\,x)$ (2). Since $\varphi(0) = 0$ then $A = 0$, so $\varphi(x) = B\sin(\sqrt{\lambda}\,x)$. Then $\varphi'(x) = B\sqrt{\lambda}\cos(\sqrt{\lambda}\,x)$ (2), and $\varphi'(1) + \varphi(1) = 0$ implies that $B\sqrt{\lambda}\cos(\sqrt{\lambda}) + B\sin(\sqrt{\lambda}) = 0$. (2) So, for nontrivial solutions ($B \neq 0$), we must have $\sqrt{\lambda}\cos(\sqrt{\lambda}) = -\sin\sqrt{\lambda}$, or (2) $\tan\sqrt{\lambda} = -\sqrt{\lambda}$.

Let $\omega_1, \omega_2, \omega_3, \dots$ be the solutions of $\tan\omega = -\omega$ (see diagram:



); Then $\sqrt{\lambda} = \omega_n$,

so $\lambda = (\omega_n)^2$ (3) are the eigenvalues, and the eigenfunctions are $\varphi_n(x) = \sin(\omega_n x)$. (2) Since $G(t) = e^{-k\lambda t}$ (2), then $G(t) = e^{-k\omega_n^2 t}$, and separated solutions are $\sin(\omega_n x)\,e^{-k\omega_n^2 t}$.

Taking $\boxed{u(x,t) = \sum_{n=1}^{\infty} B_n \sin(\omega_n x)\,e^{-k\omega_n^2 t}}$ (2), we see that $u(x,0) = f(x)$ if $f(x) = \sum_{n=1}^{\infty} B_n \sin(\omega_n x)$. By orthogonality, for each m,

$\int_0^1 f(x)\sin(\omega_m x)\,dx = \int_0^1 B_m \sin^2(\omega_m x)\,dx$. So

$$\boxed{B_m = \frac{\int_0^1 f(x)\sin(\omega_m x)\,dx}{\int_0^1 \sin^2(\omega_m x)\,dx}}$$

(4)