EXAM 3
Math 4163
4-26-13

Name key

Instructions *Work all of the following problems in the space provided. If there is not enough room, you may write on the back sides of the pages. Give thorough explanations to receive full credit.*

1. (*25 points*) Solve Laplace's equation,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0,$$

on the box $0 < x < L$, $0 < y < W$, $0 < z < H$, with the boundary conditions

$$\begin{array}{lll} u(0,y,z) = 0 & u(x,0,z) = 0 & u(x,y,0) = 0 \\ u(L,y,z) = 0 & u(x,W,z) = 0 & u(x,y,H) = \alpha(x,y). \end{array}$$

You may assume that the product solutions $u(x,y,z) = \phi(x,y)h(z)$ satisfy

$$\frac{d^2 h}{dz^2} = \lambda h, \qquad \nabla^2 \phi = -\lambda \phi,$$

and you do not need to rederive the formulas for the eigenvalues λ and eigenfunctions ϕ, if you remember them.

For Dirichlet boundary conditions ($\phi(0,y) = \phi(L,y) = \phi(x,0) = \phi(x,W) = 0$), the eigenfunctions for $\nabla^2\phi = -\lambda\phi$ are $\phi_{mn}(x,y) = \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi y}{W}\right)$, with eigenvalues $\lambda_{mn} = \sqrt{\left(\frac{m\pi}{L}\right)^2 + \left(\frac{n\pi}{W}\right)^2}$; $m = 1,2,3,\ldots$; $n = 1,2,3,\ldots$.

The solutions of $\frac{d^2h}{dz^2} = \lambda_{mn} h$ are $h(z) = A\cosh(\sqrt{\lambda_{mn}}\, z) + B\sinh(\sqrt{\lambda_{mn}}\, z)$,

so take $u(x,y,z) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi y}{W}\right)\left[A_{mn}\cosh(\sqrt{\lambda_{mn}}\, z) + B_{mn}\sinh(\sqrt{\lambda_{mn}}\, z)\right]$.

Putting $z=0$ we get $0 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi y}{W}\right) A_{mn}$, so $A_{mn} = 0$ for all m, n.

Hence $\boxed{u(x,y,z) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi y}{W}\right) B_{mn}\sinh(\sqrt{\lambda_{mn}}\, z).}$

Putting $z=H$ we get $\alpha(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi y}{L}\right) B_{mn}\sinh(\sqrt{\lambda_{mn}}\, H)$,

so by orthogonality we get $\int_0^L\int_0^W \alpha(x,y)\sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi y}{W}\right)dx\,dy = B_{mn}\sinh(\sqrt{\lambda_{mn}}\, H)\int_0^L\int_0^W \sin^2\left(\frac{m\pi x}{L}\right)\sin^2\left(\frac{n\pi y}{W}\right)dx\,dy$

or $\boxed{B_{mn} = \frac{4}{LW}\cdot\frac{1}{\sinh(\sqrt{\lambda_{mn}}\, H)}\int_0^L\int_0^W \alpha(x,y)\sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi y}{W}\right)dx\,dy}$

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2. (*points*) Consider the equation
25

$$\frac{\partial^2 u}{\partial t^2} = 3\frac{\partial^2 u}{\partial x^2} + 4\frac{\partial^2 u}{\partial y^2}$$

on the rectangle $0 < x < 1$, $0 < y < 1$, with Dirichlet boundary conditions $u = 0$ on all four sides of the rectangle. Find all separated solutions of the form $u(x, y, t) = f(x)g(y)h(t)$.

Putting $u = f(x)g(y)h(t)$ into the equation gives

$$f \cdot g \cdot \frac{d^2h}{dt^2} = 3\frac{d^2f}{dx^2}\cdot g \cdot h + 4 f \cdot \frac{d^2g}{dy^2} \cdot h .$$

Dividing by $f \cdot g \cdot h$ we get $\quad \dfrac{\frac{d^2h}{dt^2}}{h} = \dfrac{3\frac{d^2f}{dx^2}}{f} + \dfrac{4 \cdot \frac{d^2g}{dy^2}}{g}$, 4

and since the variables are separated, we can set both sides equal to a constant, say $-\lambda$. Then $\frac{d^2h}{dt^2} = -\lambda h$ 2 and

$$\frac{3\frac{d^2f}{dx^2}}{f} + \frac{4\frac{d^2g}{dy^2}}{g} = -\lambda \; (2) \Rightarrow \frac{3\frac{d^2f}{dx^2}}{f} = -\frac{4\frac{d^2g}{dy^2}}{g} - \lambda \; (2)$$

. Again the variables are separated so we can set both sides equal to a constant, say $-\mu$.

So $\frac{d^2f}{dx^2} = -\frac{\mu}{3} f$ 2 and $\dfrac{-4\frac{d^2g}{dy^2}}{g} - \lambda = -\mu \Rightarrow \dfrac{d^2g}{dy^2} = \dfrac{-(\lambda-\mu)}{4} g$. 3

Since we have Dirichlet boundary conditions in both x and y, 2

The solutions are given by: $\frac{\mu}{3} = m^2\pi^2$, $f(x) = \sin(m\pi x)$; $m = 1, 2, 3, \ldots$

and $\frac{\lambda-\mu}{4} = n^2\pi^2$, $g(y) = \sin(n\pi y)$; 2 $n = 1, 2, 3, \ldots$

So $\mu = 3m^2\pi^2$, $\lambda = 4n^2\pi^2 + \mu = 4n^2\pi^2 + 3m^2\pi^2$. 2 Since $\lambda > 0$,

then $h(t) = A\cos(\sqrt{\lambda}\, t) + B\sin(\sqrt{\lambda}\, t)$. 2 Hence separated solutions are 2

$$\boxed{\begin{gathered} u(x,y,t) = f(x)g(y)h(t) = \sin(m\pi x)\sin(n\pi y)\left[A\cos(\sqrt{\lambda_{mn}}\, t) + B\sin(\sqrt{\lambda_{mn}}\, t)\right] \\ \text{where } \lambda_{mn} = 3m^2\pi^2 + 4n^2\pi^2 \; ; \; m = 1, 2, 3, \ldots \text{ and } n = 1, 2, 3, \ldots \end{gathered}}$$

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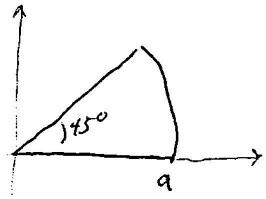
3. *(30 points)* Shown in the diagram is a sector of a circle of radius a with angle 45^o (or $\pi/4$ radians). Solve the heat equation

$$\frac{\partial u}{\partial t} = k\nabla^2 u = k\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right]$$



on this sector, with boundary conditions

$$\frac{\partial u}{\partial \theta}(r,0,t) = 0, \quad \frac{\partial u}{\partial \theta}(r,\pi/4,t) = 0, \quad u(a,\theta,t) = 0$$

and initial condition

$$u(r,\theta,0) = \alpha(r,\theta).$$

You may assume that the product solutions $u(r,\theta,t) = f(r)g(\theta)h(t)$ satisfy

$$\frac{dh}{dt} = -\lambda k h, \quad \frac{d^2 g}{d\theta^2} = -\mu g, \quad r^2\frac{d^2 f}{dr^2} + r\frac{df}{dr} + (\lambda r^2 - \mu)f = 0,$$

with $\lambda > 0$.

The boundary conditions give $\frac{dg}{d\theta}(0) = \frac{dg}{d\theta}(\frac{\pi}{4}) = 0$, (2) so the solutions of the equation for g are $g(\theta) = \cos\left(\frac{m\pi\theta}{\pi/4}\right) = \cos(4m\theta)$ (2) $(m = 0, 1, 2, 3, \dots)$ with eigenvalues $\mu = +(4m)^2$. (2) Putting $z = \sqrt{\lambda} r$ in the equation for f gives $z^2\frac{d^2\tilde{f}}{dz^2} + z\frac{d\tilde{f}}{dz} + (z^2 - (4m)^2)\tilde{f} = 0$, where $\tilde{f}(\sqrt{\lambda} r) = f(r)$.

This is Bessel's equation of order $4m$, so the general solution is $\tilde{f}(z) = AJ_{4m}(z) + BY_{4m}(z)$, but for solutions which are finite at $z=0$ we must take $B=0$, so $\tilde{f}(z) = AJ_{4m}(z)$ (2) and $f(r) = AJ_{4m}(\sqrt{\lambda} r)$. (2)

From $u(a,\theta,t) = 0$ we get $J_{4m}(\sqrt{\lambda} a) = 0$, so $\sqrt{\lambda} a = z_{4m,n}$ (2) where $z_{4m,n}$ $(n = 1, 2, 3, \dots)$ are the zeros of J_{4m}. Hence $\boxed{\lambda = \lambda_{mn} = \left(\frac{z_{4m,n}}{a}\right)^2.}$ (3)

From the equation for h we get $h(t) = Ae^{-k\lambda_{mn} t}$. (3) So take

$$\boxed{u(r,\theta,t) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_{4m}(\sqrt{\lambda_{mn}}\, r)\cos(4m\theta)\, e^{-k\lambda_{mn} t}\cdot A_{mn}}$$ (3)

Putting $t=0$ gives $\alpha(r,\theta) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} A_{mn} J_{4m}(\sqrt{\lambda_{mn}}\, r)\cos(4m\theta)$, (3) and from orthogonality we get

(6)
$$\boxed{A_{mn} = \frac{\int_0^{\pi/4}\int_0^a \alpha(r,\theta) J_{4m}(\sqrt{\lambda_{mn}}\, r)\cos(4m\theta)\, r\,dr\,d\theta}{\int_0^{\pi/4}\int_0^a J_{4m}(\sqrt{\lambda_{mn}}\, r)^2\cos^2(4m\theta)\, r\,dr\,d\theta}}$$

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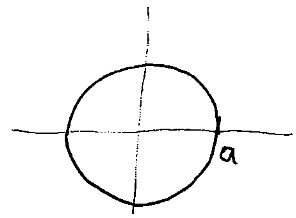
4. *(20 points)* Consider the wave equation for circularly symmetric functions,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u = c^2 \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r}\right),$$



on the circular region $0 < r < a$, with boundary condition

$$\frac{\partial u}{\partial r}(a,t) = 0.$$

Find all separated solutions of the form $u(r,t) = f(r)h(t)$, where

$$\frac{d^2 h}{dt^2} = -\lambda c^2 h, \quad r^2 \frac{d^2 f}{dr^2} + r\frac{df}{dr} + \lambda r^2 f = 0,$$

and $\lambda > 0$.

Putting $z = \sqrt{\lambda} r$, we get, for the function $\tilde{f}(z)$ related to $f(r)$ by $\tilde{f}(\sqrt{\lambda} r) = f(r)$, the equation

$$z^2 \frac{d^2 \tilde{f}}{dz^2} + z \frac{d\tilde{f}}{dz} + z^2 \tilde{f} = 0,$$

which is Bessel's equation of order zero and therefore has the general solution $\tilde{f}(z) = A J_0(z) + B Y_0(z)$. (2) Since we require $f(r)$ to be finite at $r=0$, we must take $B=0$, so $\tilde{f}(z) = A J_0(z)$ (2) and so $f(r) = A J_0(\sqrt{\lambda} r)$. (2)

From $\frac{du}{dr}(a,t) = 0$ we get $\frac{df}{dr}(a) = 0$. (2) Since $f'(r) = A\sqrt{\lambda} J_0'(\sqrt{\lambda} r)$, (2) this means that $J_0'(\sqrt{\lambda} a) = 0$. So, if $\boxed{w_{0n} \text{ stands for the } n^{th} \text{ zero of } J_0'(z)}$, (2) we must have $\sqrt{\lambda} a = w_{0n}$ for some $n = 1, 2, 3, \ldots$

or $\boxed{\lambda_n = \left(\frac{w_{0n}}{a}\right)^2}$. (2)

From the equation for h we get (2)

$$h(t) = A \cos(c\sqrt{\lambda_n} t) + B \sin(c\sqrt{\lambda_n} t),$$

so

$$\boxed{u(r,t) = J_0(\sqrt{\lambda_n} r)\{A \cos(c\sqrt{\lambda_n} t) + B \sin(c\sqrt{\lambda_n} t)\} \quad n = 1, 2, 3, \ldots}$$ (2)