FINAL EXAM
Math 4163
4-26-13

Name Key

Instructions *Work all of the following problems in the space provided. If there is not enough room, you may write on the back sides of the pages. Give thorough explanations to receive full credit.*

1. *(20 points)* Consider the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},$$

for $0 < x < 2$, $t > 0$, subject to the boundary conditions

$$\frac{\partial u}{\partial x}(0,t) = 0 \quad \text{and} \quad \frac{\partial u}{\partial x}(2,t) = 0.$$

a) Find, in series form, the solution $u(x,t)$ satisfying the initial condition

$$u(x,0) = \begin{cases} 0 & \text{for } 0 \le x \le 1 \\ 10 & \text{for } 1 < x \le 2. \end{cases}$$

Separating $u(x,t) = \varphi(x)\, h(t)$ gives $\varphi''(x) = -\lambda \varphi(x)$ (2) and $h'(t) = -\lambda h(t)$ (2)

with $\varphi'(0) = \varphi'(2) = 0$. So $\lambda = \left(\frac{n\pi}{2}\right)^2$ and $\varphi_n(x) = \cos\left(\frac{n\pi x}{2}\right)$ for $n = 0, 1, 2, \ldots$

and $h(t) = e^{-(n\pi/2)^2 t}$ (2)

Take $\boxed{u(x,t) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{2}\right) e^{-\left(\frac{n\pi}{2}\right)^2 t}}$ (2)

Putting $t=0$ gives $f(x) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{2}\right)$ (1) where $f(x) = \begin{cases} 0, & 0 \le x \le 1 \\ 10, & 1 < x < 2. \end{cases}$

So $A_0 = \frac{1}{2}\int_0^2 f(x)\,dx$ (1) $= \frac{1}{2}\int_1^2 10\,dx$ (1) $= \frac{1}{2}\left[10x\right]_1^2 = \frac{20-10}{2} = \boxed{5}$, (1)

and for $n = 1, 2, 3, \ldots$

$$A_n = \frac{2}{2}\int_0^2 f(x)\cos\left(\frac{n\pi x}{2}\right)dx \;(1)\; = \int_1^2 10\cos\left(\frac{n\pi x}{2}\right)dx \;(1)\; =$$

$$= 10\cdot\left(\frac{2}{n\pi}\right)\left[\sin\left(\frac{n\pi x}{2}\right)\right]_{x=1}^{x=2} \;(1)\; = \frac{20}{n\pi}\left[\sin(n\pi) - \sin\left(\frac{n\pi}{2}\right)\right] \;(1)\; = \boxed{\frac{-20\sin\left(\frac{n\pi}{2}\right)}{n\pi}} \;(1)$$

b) Write out explicitly the first three non-zero terms of the series for $u(x,t)$.

$$u(x,t) = A_0 + A_1\cos\left(\frac{\pi x}{2}\right)e^{-(\pi/2)^2 t} + A_2\cos(\pi x)e^{-(\pi)^2 t} + A_3\cos\left(\frac{3\pi x}{2}\right)e^{-(3\pi/2)^2 t} + \ldots$$

Since $A_0 = 5$, $A_1 = \frac{-20}{\pi}$, $A_2 = 0$, $A_3 = \frac{+20}{3\pi}$, ... the first three

non-zero terms are

$$\boxed{u(x,t) = 5 \;(1)\; - \frac{20}{\pi}\cos\left(\frac{\pi x}{2}\right)e^{-\left(\frac{\pi}{2}\right)^2 t} \;(1)\; + \frac{20}{3\pi}\cos\left(\frac{3\pi x}{2}\right)e^{-\left(\frac{3\pi}{2}\right)^2 t} \;(1)\; + \ldots}$$

2. *(20 points)* Consider the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

for $u(x,t)$ on $0 < x < 1$, $t > 0$, with boundary conditions

$$u(0,t) = 0 \quad \text{and} \quad \frac{\partial u}{\partial x}(1,t) - 2u(1,t) = 0$$

for $t \geq 0$. Separated solutions are of the form $u(x,t) = \phi(x)G(t)$ where

$$\frac{dG}{dt} = -\lambda G, \qquad \phi''(x) = -\lambda\phi.$$

[8] *a)* Find an equation for the positive eigenvalues ($\lambda > 0$) and solve it graphically, using the graph below.

For $\lambda > 0$, $\varphi(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x)$. 1

We have $\varphi(0) = 0$, so $A = 0$ and

$\varphi(x) = B\sin(\sqrt{\lambda}x)$. 1

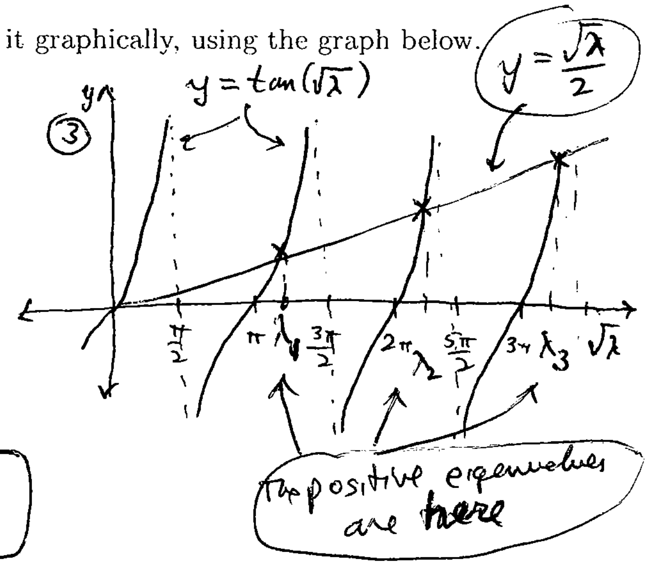
We also have $\varphi'(1) - 2\varphi(1) = 0$, 1 and since

$\varphi'(x) = B\sqrt{\lambda}\cos(\sqrt{\lambda}x)$, 1 This means

$B\sqrt{\lambda}\cos(\sqrt{\lambda}) - 2B\sin(\sqrt{\lambda}) = 0$, 1

or $\sqrt{\lambda}\cos(\sqrt{\lambda}) = 2\sin(\sqrt{\lambda})$, or

$\frac{\sqrt{\lambda}}{2} = \frac{\sin(\sqrt{\lambda})}{\cos(\sqrt{\lambda})}$, or $\boxed{\tan(\sqrt{\lambda}) = \frac{\sqrt{\lambda}}{2}}$



b) If $\lambda > 0$, what is the limit of $G(t)$ as $t \to \infty$? Briefly explain your answer.

[2] $G(t) = Ce^{-\lambda t}$, so $\lim_{t\to\infty} G(t) = 0$.

[8] *c)* Find an equation for the negative eigenvalue(s) ($\lambda < 0$) and solve it graphically, using the graph below.

For $\lambda < 0$, $\varphi(x) = A\cosh(\sqrt{|\lambda|}x) + B\sinh(\sqrt{|\lambda|}x)$. 1

Again $\varphi(0) = 0 \Rightarrow A = 0 \Rightarrow$

$\varphi(x) = B\sinh(\sqrt{|\lambda|}x)$ and 1

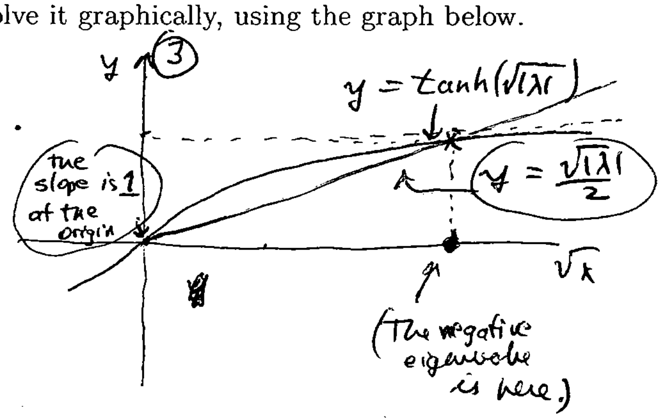
$\varphi'(x) = B\sqrt{|\lambda|}\cosh(\sqrt{|\lambda|}x)$. 1

So $\varphi'(1) - 2\varphi(1) = 0 \Rightarrow$ 1

$B\sqrt{|\lambda|}\cosh(\sqrt{|\lambda|}) - 2B\sinh(\sqrt{|\lambda|}) = 0 \Rightarrow$ 1

$\sqrt{|\lambda|}\cosh(\sqrt{|\lambda|}) = 2\sinh(\sqrt{|\lambda|})$, or

$\frac{\sqrt{|\lambda|}}{2} = \frac{\sinh(\sqrt{|\lambda|})}{\cosh(\sqrt{|\lambda|})}$, or $\boxed{\tanh(\sqrt{|\lambda|}) = \frac{\sqrt{|\lambda|}}{2}}$



d) If $\lambda < 0$, what is the limit of $G(t)$ as $t \to \infty$? Briefly explain your answer.

[2] $G(t) = Ce^{-\lambda t} = Ce^{|\lambda|t}$, so $\lim_{t\to\infty} G(t) =$

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3. *(20 points)* Solve the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right).$$

for $0 < x < \pi$, $0 < y < \pi$, $t > 0$, with the boundary conditions

$$u(0,y,t) = 0 \qquad u(x,0,t) = 0$$
$$u(\pi,y,t) = 0 \qquad u(x,\pi,t) = 0$$

and the initial conditions

$$u(x,y,0) = 1$$
$$\frac{\partial u}{\partial t}(x,y,0) = 0.$$

You do not need to rederive the formulas for the eigenvalues λ and eigenfunctions ϕ, if you remember them.

Be as explicit as you can in giving the formulas for the coefficients in the series for u.

(2)

Separating $u(x,y,t) = \varphi(x,y)\, h(t)$, we get $\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = -\lambda\varphi$ and $h''(t) = -c^2\lambda h$ with $\varphi = 0$ on the boundary of the rectangle $0 \le x \le \pi$, $0 \le y \le \pi$. (2)

The eigenfunctions for φ are $\varphi_{mn}(x) = \sin(mx)\sin(ny)$, with corresponding eigenvalues $\boxed{\lambda_{mn} = m^2 + n^2}$ (2) Then ~~also~~

(2) $h(t) = A\cos(c\sqrt{\lambda_{mn}}\,t) + B\sin(c\sqrt{\lambda_{mn}}\,t)$. So we take

(2) $$\boxed{u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \sin(mx)\sin(ny)\left[A_{mn}\cos(c\sqrt{\lambda_{mn}}\,t) + B_{mn}\sin(c\sqrt{\lambda_{mn}}\,t)\right].}$$

Taking the derivative w.r.t. t gives

(2) $$\frac{\partial u}{\partial t} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \sin(mx)\sin(ny)\left[-c\sqrt{\lambda_{mn}}\,A_{mn}\sin(c\sqrt{\lambda_{mn}}\,t) + c\sqrt{\lambda_{mn}}\,B_{mn}\cos(c\sqrt{\lambda_{mn}}\,t)\right]$$

and putting $t=0$ gives (2)

$$0 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \sin(mx)\sin(ny)\left[c\sqrt{\lambda_{mn}}\,B_{mn}\right],$$ which implies $\boxed{B_{mn} = 0 \text{ for all } m,n = 1,2,3,\dots}$

so (2) $u(x,y,0) = 1 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\sin(mx)\sin(ny)$, and hence

(1) (1)

$$\boxed{A_{mn}} = \frac{2}{\pi}\cdot\frac{2}{\pi}\int_0^{\pi}\int_0^{\pi} 1\,\sin(mx)\sin(ny)\,dx\,dy = \frac{4}{\pi^2}\int_0^{\pi}\sin(mx)\,dx\int_0^{\pi}\sin(ny)\,dy =$$

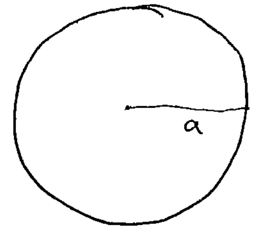
(1) (1)

$$= \frac{4}{\pi^2}\left[\frac{-\cos(mx)}{m}\right]_0^{\pi}\left[-\frac{\cos(ny)}{n}\right]_0^{\pi} = \frac{4}{\pi^2}\left[\frac{-\cos(m\pi)+1}{m}\right]\left[\frac{-\cos(n\pi)+1}{n}\right] =$$

$$= \boxed{\frac{4}{mn\pi^2}(1-\cos(m\pi))(1-\cos(n\pi)) = \begin{cases} 0 & \text{if either } m \text{ or } n \text{ is even} \\ \boxed{\frac{16}{mn\pi^2}} & \text{if } m \text{ and } n \text{ are both odd} \end{cases}}$$

4. *(20 points)* Solve the equation

$$\nabla^2 u + \frac{\partial^2 u}{\partial z^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0$$



for $0 < r < a$, $-\pi \le \theta \le \pi$, $0 < z < H$, with boundary conditions

$$u(a,\theta,z) = 0, \quad u(r,\theta,0) = 0, \quad u(r,\theta,H) = \alpha(r,\theta).$$

(periodic boundary conditions in θ)

You may assume that the product solutions $u(r,\theta,z) = f(r)g(\theta)h(z)$ satisfy

$$\frac{d^2h}{dz^2} = \lambda h, \quad \frac{d^2 g}{d\theta^2} = -\mu g, \quad r^2\frac{d^2 f}{dr^2} + r\frac{df}{dr} + (\lambda r^2 - \mu)f = 0,$$

with $\lambda > 0$.

Since g satisfies periodic boundary conditions, $g(-\pi) = g(\pi)$ and $g'(-\pi) = g'(\pi)$ (2) we have $\mu = m^2$, $m = 0, 1, 2, 3, \ldots$ and $g(\theta) = A\cos m\theta + B\sin m\theta$ (2)

The equation for f becomes $r^2 f''(r) + r f'(r) + (\lambda r^2 - m^2) f = 0$.

Putting $\tilde f(z) = f(r)$ with $z = \sqrt{\lambda}\, r$ gives $z^2 \tilde f''(z) + z\tilde f'(z) + (z^2 - m^2)\tilde f = 0$, which is Bessel's equation of order m, with general solution $\tilde f(z) = A J_m(z) + B Y_m(z)$. (2) Since f, and also $\tilde f$, are finite at $r = z = 0$, we have $B = 0$. So $\tilde f(z) = A J_m(z)$ and $f(r) = A J_m(\sqrt{\lambda}\, r)$. (2)

Since $f(a) = 0$ then $J_m(\sqrt{\lambda}\, a) = 0$, so $\sqrt{\lambda}\, a = z_{mn}$ where $\{z_{mn}\}_{n=1,2,3,\ldots}$ (2) are the zeroes of $J_m(z)$. So $\boxed{\lambda_{mn} = \left(\frac{z_{mn}}{a}\right)^2}$ and $f(r) = A J_m(\sqrt{\lambda_{mn}}\, r)$.

Hence $h(z) = C\cosh(\sqrt{\lambda_{mn}}\, z) + D\sinh(\sqrt{\lambda_{mn}}\, z)$ (2) and we take

$$u(r,\theta,z) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(\sqrt{\lambda_{mn}}\, r)\left[A_{mn}\cos m\theta + B_{mn}\sin m\theta\right]\left[C_{mn}\cosh(\sqrt{\lambda_{mn}}\, z) + D_{mn}\sinh(\sqrt{\lambda_{mn}}\, z)\right]$$

Since $u(r,\theta,0) = 0$ we get $C_{mn} = 0$ for all m, n, so (2)

$$\boxed{u(r,\theta,z) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(\sqrt{\lambda_{mn}}\, r)\left[\tilde A_{mn}\cos m\theta + \tilde B_{mn}\sin m\theta\right]\sinh(\sqrt{\lambda_{mn}}\, z)}$$

(where $\tilde A_{mn} = A_{mn}D_{mn}$, $\tilde B_{mn} = B_{mn}D_{mn}$). Putting $z = H$ gives (2)

$$\alpha(r,\theta) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(\sqrt{\lambda_{mn}}\, r)\sinh(\sqrt{\lambda_{mn}}\, H)\left[\tilde A_{mn}\cos m\theta + \tilde B_{mn}\sin m\theta\right]$$

so
$$\tilde A_{mn} = \frac{1}{\sinh(\sqrt{\lambda_{mn}}\, H)}\,\frac{\int_{-\pi}^{\pi}\int_0^a \alpha(r,\theta) J_m(\sqrt{\lambda_{mn}}\, r)(\cos m\theta)\, r\,dr\,d\theta}{\int_{-\pi}^{\pi}\int_0^a J_m(\sqrt{\lambda_{mn}}\, r)^2(\cos^2 m\theta)\, r\,dr\,d\theta}$$ (2)

and $\tilde B_{mn}$ is given by the same formula, except with "$\cos m\theta$" replaced by "$\sin m\theta$" (2)

5. *(20 points)* Solve the inhomogeneous heat equation

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + 1$$

for $0 < x < \pi$, $t > 0$, with homogeneous boundary conditions

$$v(0,t) = 0 \quad \text{and} \quad v(\pi,t) = 0,$$

and initial data

$$v(x,0) = 0.$$

Use the method of eigenfunction expansions. Write the solution as explicitly as you can (the answer should be a series in which the terms are elementary functions of x and t).

The eigenvalue problem for homogeneous boundary conditions is $\varphi''(x) = -\lambda\varphi(x)$ with $\varphi(0)=\varphi(\pi)=0$, so the eigenfunctions are $\varphi_n(x) = \sin(nx)$ with eigenvalues $\lambda = n^2$ $(n=1,2,3,\dots)$. 2

Therefore we take $v(x,t) = \sum_{n=1}^{\infty} a_n(t)\sin(nx)$. 2

Putting into $v_t = v_{xx}+1$ gives 2

$$\sum_{n=1}^{\infty} \frac{da_n}{dt}\sin(nx) = \left[\sum_{n=1}^{\infty} a_n(-n^2\sin nx)\right] + 1 = \sum_{n=1}^{\infty} a_n(-n^2\sin nx) + \sum_{n=1}^{\infty} b_n \sin nx$$ 2

where $1 = \sum_{n=1}^{\infty} b_n \sin nx$, so $b_n = \frac{2}{\pi}\int_0^{\pi} 1\cdot\sin nx\,dx = \left[\left(\frac{2}{\pi}\right)\frac{-\cos(nx)}{n}\right]_0^{\pi}$ 2

$$= \frac{2}{\pi n}\left[-\cos n\pi + 1\right] = \frac{2(1-\cos n\pi)}{\pi n} \text{ for } n=1,2,3,\dots$$

So $\sum_{n=1}^{\infty}\left(\frac{da_n}{dt} + n^2 a_n - b_n\right)\sin(nx) = 0$, so $\frac{da_n}{dt} + n^2 a_n = b_n$ for $n=1,2,3,\dots$ 2

Multiplying by e^{n^2t} gives $e^{n^2t}\frac{da_n}{dt} + n^2 e^{n^2t}a_n = e^{n^2t}b_n$, or 1

$\frac{d}{dt}\left(e^{n^2t}a_n\right) = e^{n^2t}b_n$. 1 Integrating from 0 to t gives $\left[e^{n^2t}a_n(t)\right]_0^t = \int_0^t e^{n^2t}b_n\,dt$ 1

$$= b_n\int_0^t e^{n^2t}\,dt = b_n\left[\frac{e^{n^2t}}{n^2}\right]_0^t = b_n\left[\frac{e^{n^2t}-1}{n^2}\right].$$ 1 So

$$e^{n^2t}a_n(t) - e^0 a_n(0) = \frac{b_n}{n^2}\left[e^{n^2t}-1\right].$$ 1 Since $v(x,0)=0$, then

$0 = \sum_{n=1}^{\infty} a_n(0)\sin nx$, so $a_n(0)=0$. 2 So $e^{n^2t}a_n(t) = \frac{b_n}{n^2}\left[e^{n^2t}-1\right]$, or

$a_n(t) = \frac{b_n}{n^2}\left[1-e^{-n^2t}\right] = \frac{2(1-\cos n\pi)}{\pi n^3}\left[1-e^{-n^2t}\right]$. 1 Hence $\boxed{v(x,t) = \sum_{n=1}^{\infty}\frac{2(1-\cos n\pi)}{\pi n^3}\left[1-e^{-n^2t}\right]\sin nx}$