Math 4163 — Spring 2013
Review for Final Exam

The final exam is comprehensive. To review for it, you can use the review sheets for the first three exams to study the material in sections 2.3, 2.4, 2.5, 3.2, 3.3, 4.4, 5.3, 5.8, 7.2, 7.3, 7.4, 7.6, 7.7, and 7.8 of the text. There will also be some questions on the material covered since the third exam, which is in sections 7.10, 8.2, and 8.3 of the text. See below for a guide to what material in these latter three sections you should look at.

7.10. Spherical problems and Legendre polynomials. This section begins with a discussion of the solution of the three-dimensional wave equation $\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$ for given boundary and initial values on the sphere, and concludes with a discussion of the solution of Laplace's equation $\nabla^2 u = 0$ for given boundary values on the sphere. In class we only talked about Laplace's equation on the sphere. But when reading the text, to be able to follow what the author is doing, you'll probably want to read about both the wave equation and Laplace's equation.

Subsection 7.10.2 explains what happens when you express the wave equation in spherical coordinates, and then look for separated solutions by putting $u(\rho, \theta, \phi, t) = f(\rho)q(\theta)g(\phi)h(t)$ and separating variables. You will get four ordinary differential equations, one for each of the functions f, q, g, and h. The equation for $h(t)$, which is $d^2h/dt^2 = -\lambda c^2 h$, and the equation for $q(\theta)$, which is $d^2q/d\theta^2 = -\sigma q$, are familiar to you already. (Note: the text doesn't bother to write out the eigenvalue problem for $q(\theta)$, since we've seen it before many times. When I wrote it out here I used the letter σ for the eigenvalue, because the letters λ or μ are already used for other eigenvalues in this problem.) The equation for $q(\theta)$ has periodic boundary conditions, so as we already know, its eigenfunctions are given by $\sigma = m^2$, $m = 0, 1, 2, 3, \ldots$, and the corresponding eigenfunctions are 1 (for $m = 0$) and $\cos m\theta$ and $\sin m\theta$ (for $m = 1, 2, 3, \ldots$). The equation for $g(\phi)$ is given by (7.10.11), and it is used to determine the eigenvalues μ. Finally, the equation for $f(\rho)$ is given by (7.10.10), and it is used to determine the eigenvalues λ.

Subsection 7.10.3 explains what the solutions of the equation (7.10.11) for $g(\phi)$ look like, and what the eigenvalues μ are. It turns out that the eigenvalues μ are given by $\mu = n(n+1)$ for $n = m, m+1, m+2, \ldots$. The corresponding eigenfunctions are given by

$$g(\phi) = P_n^m(\cos\phi),$$

where P_n^m are functions called associated Legendre functions, and are given by the formulas in (7.10.19) and (7.10.18) (see also (7.10.17) for some examples). You do not need to know any of the definitions or formulas for associated Legendre functions, or how to find them. It's enough just to know that they are the solutions of (7.10.11) corresponding to the eigenvalue $\mu = n(n+1)$. You do not need to remember equation (7.10.11), either. I'll provide it on the test if it's necessary to use it.

Subsection 7.10.4 gives formulas for the solutions of the equation (7.10.10) for $f(\rho)$. You should read this subsection just to help get a better picture of what's going on, but I will not expect you know anything from this subsection on the exam. (Notice: the book only skims over the fact that equation (7.10.10), together with the given boundary conditions for $f(\rho)$ at $\rho = 0$ and $\rho = a$, will determine the possible eigenvalues λ for the problem. For each choice of m and n, there are infinitely many possible eigenvalues λ, so properly speaking they should be labeled λ_{mnk} for $k = 1, 2, 3, \ldots$)

Subsection 7.10.6 treats the problem of Laplace's equation on a sphere. This is simpler than the problem for the wave equation, because there is no function $h(t)$ and because there is no eigenvalue λ in the equation for $f(\rho)$. In fact, the equation for $f(\rho)$ can be simply solved, and its solutions are given by $f(\rho) = \rho^n$ and $f(\rho) = \rho^{-n-1}$. We discard the latter one, because it's infinite as $\rho \to 0$. So the general solution to Laplace's equation is given by the relatively simple formula in (7.10.30). The coefficients are determined from the boundary conditions by orthogonality in the usual way. You should be familiar with the material in this subsection; I covered it in class, and there were two homework problems assigned on it.

8.2. Heat flow with sources and nonhomogeneous boundary conditions. This section explains what inhomogeneous partial differential equations and boundary conditions look like, and how to reduce them to problems with homogeneous boundary conditions. The main tool to use is the change of variables in (8.2.24) and (8.2.25).

8.3. Method of eigenfunction expansion with homogeneous boundary conditions. We went through this entire section carefully in class, including the details of the example on page 351. As mentioned in class, I don't recommend trying to remember the formula (8.3.10). Instead, just remember the procedure for solving the linear first-order ordinary differential equation (8.3.9): multiply through by the integrating factor $e^{k\lambda_n t}$, integrate both sides from 0 to t, and solve for $a_n(t)$, as done in class.