Math 4163 — Spring 2015
Assignment 5

1.(a) Determine the coefficients of the Fourier series for the function defined by $f(x) = 1 + x$ on $[-\pi, \pi]$, and sketch the function that the Fourier series converges to on $[-3\pi, 3\pi]$.

(b) Determine the coefficients of the Fourier sine series for the function defined by $f(x) = 1 + x$ on $[0, \pi]$, and sketch the function that the Fourier sine series converges to on $[-3\pi, 3\pi]$.

(c) Determine the coefficients of the Fourier cosine series for the function defined by $f(x) = 1 + x$ on $[0, \pi]$, and sketch the function that the Fourier series converges to on $[-3\pi, 3\pi]$.

2. Consider the following initial-boundary-value problem for the function $u(x, t)$ defined for $0 \leq x \leq L$ and $t \geq 0$:

$$\begin{aligned} \rho_0 u_{tt} + au &= T_0 u_{xx} \quad \text{for } 0 < x < L,\ t > 0 \\ u(0, t) &= 0 \quad \text{for } t \geq 0 \\ u(L, t) &= 0 \quad \text{for } t \geq 0 \\ u(x, 0) &= 0 \quad \text{for } 0 \leq x \leq L \\ u_t(x, 0) &= g(x) \quad \text{for } 0 \leq x \leq L, \end{aligned}$$

where a is a constant, and $f(x)$ is a known (given) function of x.

(a) Put $u(x, t) = \phi(x)h(t)$ and separate variables to get ordinary differential equations for $\phi(x)$ and $h(t)$. If you do this in the same way as the text does it for the wave equation on pages 137–138, then you should be able to get the same equation for ϕ as in (4.4.8); that is, $\phi''(x) + \lambda\phi(x) = 0$. The equation for $h(t)$ will be different than (4.4.7), however. What is it?

(b) What boundary conditions on $\phi(x)$ does the problem determine? What are the eigenvalues and eigenfunctions for the boundary-value problem for ϕ? (You can just give the answer, no need to re-derive it here.)

(c) Find the general solution of the equation for $h(t)$ that you obtained in part (a). Since it is an equation with constant coefficients, you can find the general solution using the characteristic equation. In this step, you can assume that $a > 0$. This should guarantee that the roots of the characteristic equation will be imaginary numbers, so the general solution can be written using sines and cosines. In fact, the general solution should be of the form

$$h(t) = A\cos(\omega_n t) + B\sin(\omega_n t),$$

where the numbers ω_n are constants which depend on n, called the *frequencies of vibration* of the system. Give the correct formula for the frequencies of vibration.

(d) What are the separated solutions of the partial differential equation with the boundary conditions given?

(e) Find a linear combination of the separated solutions from (d) which satisfies the given initial conditions. Write your answer as a series, and give formulas for the coefficients in the series in terms of integrals involving $g(x)$.

3. Consider the following initial-boundary-value problem for the function $u(x, t)$ defined for $0 \leq x \leq L$ and $t \geq 0$:

$$\begin{aligned} \rho_0 u_{tt} + \beta u_t &= T_0 u_{xx} \quad \text{for } 0 < x < L,\ t > 0 \\ u(0, t) &= 0 \quad \text{for } t \geq 0 \\ u(L, t) &= 0 \quad \text{for } t \geq 0 \\ u(x, 0) &= f(x) \quad \text{for } 0 \leq x \leq L \\ u_t(x, 0) &= g(x) \quad \text{for } 0 \leq x \leq L, \end{aligned}$$

where $f(x)$ and $g(x)$ are assumed to be known (given) functions of x.

(a) Put $u(x, t) = \phi(x)h(t)$ and separate variables to get ordinary differential equations for $\phi(x)$ and $h(t)$. If you do this in the same way as the text does it for the wave equation on pages 137–138, then you should be able to get the same equation for ϕ as in (4.4.8); that is, $\phi''(x) + \lambda\phi(x) = 0$. The equation for $h(t)$ will be different than (4.4.7), however. What is it?

(b) What boundary conditions on $\phi(x)$ does the problem determine? What are the eigenvalues and eigenfunctions for the boundary-value problem for ϕ? (You can just give the answer, no need to re-derive it here.)

(c) Find the general solution of the equation for $h(t)$ that you obtained in part (a). Since it is an equation with constant coefficients, you can find the general solution using the characteristic equation. In this step, you can assume that β satisfies $0 < \beta^2 < \dfrac{4\pi^2 \rho_0 T_0}{L^2}$ (and hence $\beta^2 < \dfrac{4n^2\pi^2 \rho_0 T_0}{L^2}$ for all natural numbers n, as well). That will mean that the roots of the characteristic equation will be complex numbers, with both real and imaginary parts, so the general solution can be written using exponentials, sines and cosines.

(d) What are the separated solutions of the partial differential equation with the boundary conditions given?

(e) Find a linear combination of the separated solutions from (d) which satisfies the given initial conditions. Write your answer as a series, and give formulas for the coefficients in the series in terms of integrals involving $f(x)$ and $g(x)$.