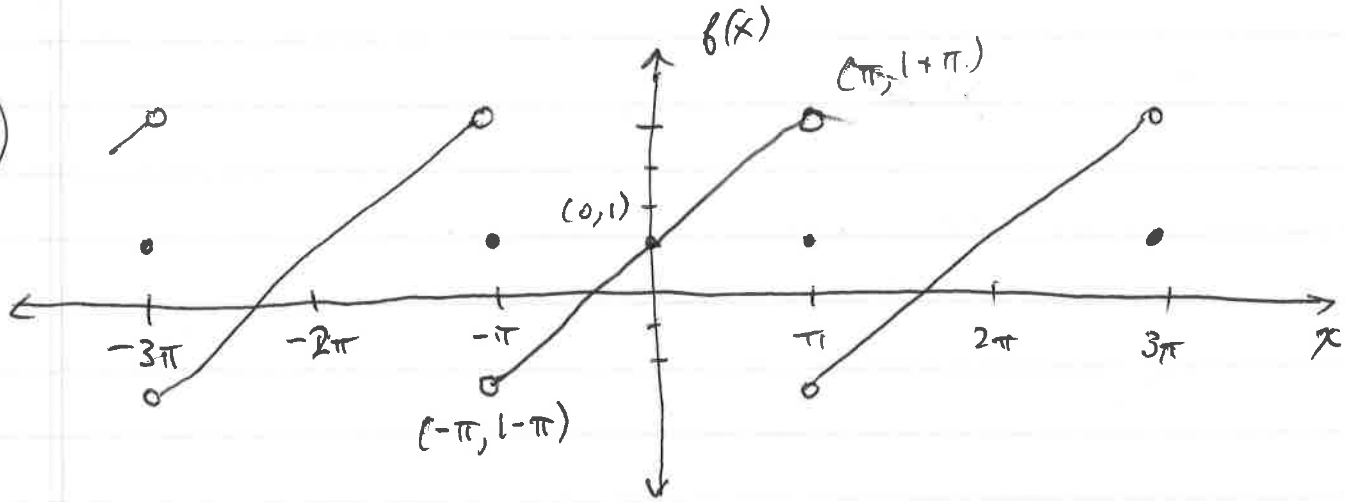
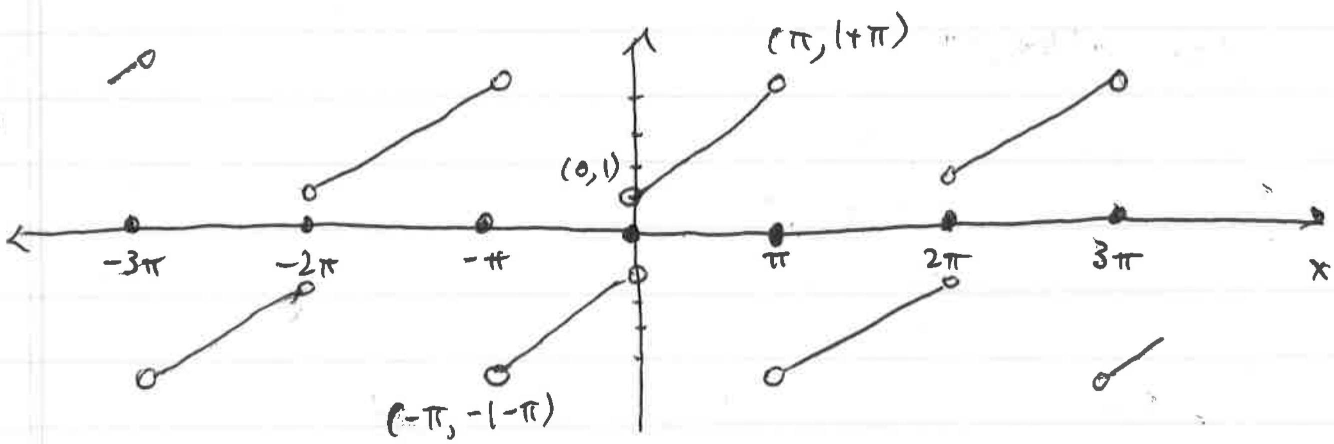
Sketches for problem 1.

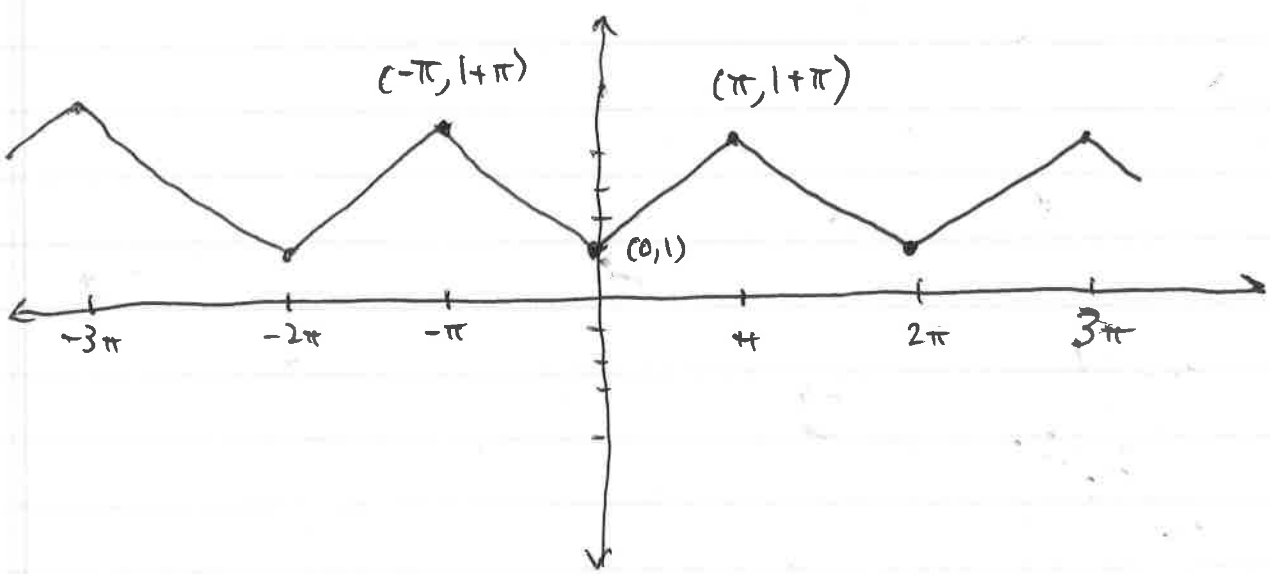
(1a)



(1b)



(1c)



Assignment 5 solutions

(For the sketches of the functions to which the Fourier series converge in problem 1, see attached sheet.)

1(a) The coefficients of the Fourier series are given by:

$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} 1 + x \; dx = 1,$$
$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (1+x)\cos(nx) \; dx = 0 \quad (\text{for } n \geq 1),$$
$$B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (1+x)\sin(nx) \; dx = \frac{-2\cos(n\pi)}{n} \quad (\text{for } n \geq 1).$$

1(b) The coefficients of the Fourier sine series are given by

$$B_n = \frac{2}{\pi}\int_{0}^{\pi} (1+x)\sin(nx) \; dx = \frac{2}{\pi n}\left(1 - (1+\pi)\cos(n\pi)\right) \quad (\text{for } n \geq 1).$$

1(c) The coefficients of the Fourier cosine series are given by

$$A_0 = \frac{1}{\pi}\int_{0}^{\pi} 1 + x \; dx = 1 + \frac{\pi}{2},$$
$$A_n = \frac{2}{\pi}\int_{0}^{\pi} (1+x)\cos(nx) \; dx = \frac{2}{\pi n^2}(\cos(n\pi) - 1) \quad (\text{for } n \geq 1).$$

2(a) A form of the ordinary differential equation for $h(t)$ is

$$\rho_0 h''(t) = -(\lambda T_0 + a)h(t).$$

2(b) The boundary conditions on $\phi(x)$ are $\phi(0) = 0$ and $\phi(L) = 0$; the eigenvalues are $\lambda = \frac{n^2\pi^2}{L^2}$ for $n = 1, 2, 3, \ldots$; and the corresponding eigenfunctions are $\phi(x) = \sin(n\pi x/L)$.

2(c) The general solution of the equation for $h(t)$, with the eigenvalues λ as given in **2(b)**, is

$$h(t) = A\cos(\omega_n t) + B\sin(\omega_n t),$$

where

$$\omega_n = \sqrt{\frac{(n\pi/L)^2 T_0 + a}{\rho_0}}.$$

2(d) The separated solutions of the partial differential equation with the given boundary conditions are

$$u(x,t) = \phi(x)h(t) = \sin\left(\frac{n\pi x}{L}\right)(A\cos(\omega_n t) + B\sin(\omega_n t)),$$

where ω_n is as given in **2(c)**.

2(e) Take

$$u(x,t) = \sum_{n=1}^{\infty}(A_n\cos(\omega_n t) + B_n\sin(\omega_n t))\sin\left(\frac{n\pi x}{L}\right).$$

The condition $u(x,0) = 0$ for $0 \leq x \leq L$ gives us that

$$0 = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right),$$

and this implies that $A_n = 0$ for every n. (Do you see why? Alternatively, you could have used the condition $u(x, 0) = 0$ back in step 1 of the method of separation of variables to say that $A = 0$ in the separated solutions individually.)

Therefore

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin(\omega_n t) \sin\left(\frac{n\pi x}{L}\right).$$

Taking the derivative with respect to t gives

$$u_t(x,t) = \sum_{n=1}^{\infty} B_n \omega_n \cos(\omega_n t) \sin\left(\frac{n\pi x}{L}\right),$$

and setting $t = 0$ and using the condition $u_t(x, 0) = g(x)$ for $0 \le x \le L$, we get

$$g(x) = \sum_{n=1}^{\infty} B_n \omega_n \sin\left(\frac{n\pi x}{L}\right) \quad \text{for } 0 \le x \le L.$$

We know well by now how to use this last equation to determine the B_n. We get

$$B_n = \frac{2}{L\omega_n} \int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right)\, dx \quad \text{for } n = 1, 2, 3, \ldots.$$

3(a) A form of the ordinary differential equation for $h(t)$ is

$$\rho_0 h''(t) + \beta h'(t) + \lambda T_0 h(t) = 0.$$

3(b) Same as in **2(b)**: the boundary conditions on $\phi(x)$ are $\phi(0) = 0$ and $\phi(L) = 0$; the eigenvalues are $\lambda = \dfrac{n^2\pi^2}{L^2}$ for $n = 1, 2, 3, \ldots$; and the corresponding eigenfunctions are $\phi(x) = \sin(n\pi x/L)$.

3(c) As we showed in class, the solution of the ordinary differential equation for $h(t)$, obtained by looking for solutions of the form e^{rt}, and taking λ as in **3(b)**, is

$$h(t) = Ae^{-\beta t/(2\rho_0)} \cos(\omega_n t) + Be^{-\beta t/(2\rho_0)} \sin(\omega_n t),$$

where

$$\omega_n = \sqrt{\frac{n^2\pi^2 T_0}{L^2 \rho_0} - \frac{\beta^2}{4\rho_0^2}}.$$

3(d) The separated solutions of the partial differential equation with the given boundary conditions are

$$u(x,t) = \phi(x)h(t) = \sin\left(\frac{n\pi x}{L}\right) e^{-\beta t/(2\rho_0)}(A\cos(\omega_n t) + B\sin(\omega_n t)),$$

where ω_n is as given in **3(c)**.

3(e) Take

$$u(x,t) = \sum_{n=1}^{\infty} e^{-\beta t/(2\rho_0)}(A_n \cos(\omega_n t) + B_n \sin(\omega_n t)) \sin\left(\frac{n\pi x}{L}\right).$$

The condition $u(x, 0) = f(x)$ for $0 \le x \le L$ gives us that

$$f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right),$$

and this implies that

$$A_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right)\, dx$$

for $n = 1, 2, 3, \ldots$.

Also, taking the derivative of $u(x,t)$ with respect to t gives

$$u_t(x,t) = \sum_{n=1}^{\infty} \left[\frac{-\beta}{2\rho_0} e^{-\beta t/(2\rho_0)}(A_n \cos(\omega_n t) + B_n \sin(\omega_n t)) + e^{-\beta t/(2\rho_0)}(-A_n\omega_n \sin(\omega_n t) + B_n\omega_n \cos(\omega_n t))\right] \sin\left(\frac{n\pi x}{L}\right).$$

Setting $t = 0$ and using the condition $u_t(x,0) = g(x)$ for $0 \le x \le L$, we get

$$g(x) = \sum_{n=1}^{\infty} \left(\frac{-\beta A_n}{2\rho_0} + B_n\omega_n\right) \sin\left(\frac{n\pi x}{L}\right) \quad \text{for } 0 \le x \le L.$$

Multiplying by $\sin(m\pi x/L)$ and integrating from $x = 0$ to $x = L$, and using orthogonality, we get,

$$\int_0^L g(x) \sin\left(\frac{m\pi x}{L}\right)\, dx = \left(\frac{-\beta A_m}{2\rho_0} + B_m\omega_m\right) \frac{L}{2}.$$

Since we already know what A_m is from above, we can use this equation to solve for B_m. Doing so, and replacing m by n (since m stands for an arbitrary number anyway), we get

$$\begin{aligned} B_n &= \frac{2}{L\omega_n} \int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right)\, dx + \frac{\beta A_n}{2\omega_n\rho_0} \\ &= \frac{2}{L\omega_n} \int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right)\, dx + \frac{\beta}{\rho_0 L\omega_n} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right)\, dx. \end{aligned}$$