Assignment 8

Consider the following boundary-value problem for the heat equation on the interval $[0,\pi]$ with time-dependent boundary conditions:

$$\begin{aligned} u_t - u_{xx} &= 0 && \text{for } 0 < x < \pi \text{ and } t > 0, \\ u(0,t) &= 0 && \text{for } t \geq 0, \\ u(\pi,t) &= \pi \cos t && \text{for } t \geq 0, \\ u(x,0) &= x && \text{for } 0 \leq x \leq \pi. \end{aligned}$$

(a) Find a simple function $r(x,t)$ satisfying the boundary conditions $r(0,t) = 0$ and $r(\pi,t) = \pi\cos t$ for $t \geq 0$.

(b) Let $v(x,t) = u(x,t) - r(x,t)$. What differential equation, boundary conditions, and initial condition does v satisfy?

(c) Substitute $v(x,t) = \sum_{n=1}^{\infty} A_n(t)\sin(nx)$ into the differential equation for v, and obtain an ordinary differential equation for $A_n(t)$. You may use that $x = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin nx$ for $0 < x < \pi$.

(d) Use the initial condition for v to determine $A_n(0)$.

(e) Use your answers to (c) and (d) to find $A_n(t)$. You may use that the general solution to the equation $\phi'(t) + n^2\phi(t) = b\sin t$ is

$$\phi(t) = \frac{b}{1+n^4}(n^2 \sin t - \cos t) + Ce^{-n^2 t}.$$

(f) Use your answers to the above to write out a formula for $u(x,t)$.