Answers to Assignment 8

(a) There are many different choices of $r(x,t)$ that would work, but probably the simplest one is

$$r(x,t) = x\cos t.$$

(b) We can put $u = v+r$ into the equations for u. From the heat equation we get $(v+r)_t-(v+r)_{xx} = 0$, so $v_t+r_t-v_{xx}-r_{xx} = 0$. Now $r_t = -x\sin t$ and $r_x = \cos t$, so $r_{xx} = 0$. Therefore we get $v_t-x\sin t-v_{xx}-0 = 0$, or

$$v_t - v_{xx} = x\sin t.$$

Also, from the boundary conditions for u we get $0 = u(0,t) = v(0,t) + r(0,t)$ and $\pi\cos t = u(\pi,t) = v(\pi,t) + r(\pi,t)$. Since $r(0,t) = 0$ and $r(\pi,t) = \pi\cos t$, these become

$$v(0,t) = 0$$
$$v(\pi,t) = 0.$$

Finally, from the initial condition for u we get $x = u(x,0) = v(x,0) + r(x,0)$. Since $r(x,0) = x$, this give

$$v(x,0) = 0.$$

(c) We have

$$v_t = \sum_{n=1}^{\infty} \frac{dA_n}{dt}\sin(nx),$$
$$v_x = \sum_{n=1}^{\infty} nA_n\cos(nx),$$
$$v_{xx} = \sum_{n=1}^{\infty} -n^2A_n\sin(nx),$$

so from $v_t - v_{xx} = x\sin t$ we get

$$\sum_{n=1}^{\infty} \frac{dA_n}{dt}\sin(nx) + \sum_{n=1}^{\infty} n^2A_n\sin(nx) = \left(\sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n}\sin(nx)\right)\sin t,$$

or

$$\sum_{n=1}^{\infty}\left(\frac{dA_n}{dt} + n^2A_n - \frac{2(-1)^{n+1}}{n}\sin t\right)\sin(nx) = 0.$$

Now we can use orthogonality of the functions $\sin(nx)$ to conclude that the only way $\sum B_n\sin(nx)$ could equal zero for all $x \in [0,\pi]$ is for all of the coefficients B_n to equal zero. So we must have that

$$\frac{dA_n}{dt} + n^2A_n - \frac{2(-1)^{n+1}}{n}\sin t = 0$$

for every $n = 1, 2, 3, \ldots$. Or, in other words, each $A_n(t)$ satisfies the ordinary differential equation

$$\frac{dA_n}{dt} + n^2A_n = \frac{2(-1)^{n+1}}{n}\sin t.$$

(d) Since $v(x,0) = 0$ from part (b), we have

$$\sum_{n=1}^{\infty} A_n(0)\sin(nx) = 0$$

for all $x \in [0, \pi]$ and therefore (by orthogonality of the functions $\sin(nx)$) we must have $A_n(0) = 0$ for all n.

(e) The function $\phi(t) = A_n(t)$ satisfies $\phi'(t) + n^2\phi(t) = b \sin t$ with $b = \dfrac{2(-1)^{n+1}}{n}$. So from the solution formula given for this ODE, we get

$$A_n(t) = \frac{2(-1)^{n+1}}{n(1+n^4)}(n^2 \sin t - \cos t) + Ce^{-n^2 t},$$

for some constant C which is independent of t (but could depend on n).

To find C, put $t = 0$ on both sides of the preceding equation to get

$$0 = A_n(0) = \frac{2(-1)^{n+1}}{n(1+n^4)}(n^2 \sin 0 - \cos 0) + Ce^0 = \frac{-2(-1)^{n+1}}{n(1+n^4)} + C,$$

so

$$C = \frac{2(-1)^{n+1}}{n(1+n^4)},$$

and therefore

$$A_n(t) = \frac{2(-1)^{n+1}}{n(1+n^4)}(n^2 \sin t - \cos t) + \frac{2(-1)^{n+1}}{n(1+n^4)}e^{-n^2 t},$$

or

$$A_n(t) = \frac{2(-1)^{n+1}}{n(1+n^4)}\left(n^2 \sin t - \cos t + e^{-n^2 t}\right).$$

(f) We have

$$\begin{aligned} u(x,t) &= v(x,t) + r(x,t) \\ &= x \cos t + \sum_{n=1}^{\infty} A_n(t) \sin(nx) \\ &= x \cos t + \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n(1+n^4)}\left(n^2 \sin t - \cos t + e^{-n^2 t}\right) \sin(nx). \end{aligned}$$