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EXAM 1
Math 4163
2-13-15

Name key

There are four problems totalling 100 points. You may write on the back sides of the pages if you need more room.

1. Consider the differential equation

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0,$$

for $0 < x < 1$, with boundary conditions

$$\frac{d\phi}{dx}(0) = 0 \quad \text{and} \quad \phi(1) = 0.$$

[12] a) *(12 points)* Check whether there are any negative eigenvalues λ. Show all work.

If $\lambda = -k^2$, the general solution of the ODE is
$(k>0)$ $\phi(x) = A\cosh kx + B\sinh kx$, 2
and $\phi'(x) = Ak\sinh kx + Bk\cosh kx$. 2
Then $\phi'(0) = 0 \Rightarrow A\cdot 0 + Bk\cdot 1 = 0$ 2 $\Rightarrow Bk = 0 \Rightarrow B = 0$ 2
So $\phi(x) = A\cosh kx$. Then $\phi(1) = 0 \Rightarrow A\cosh k = 0 \Rightarrow$ 2 $A = 0$ (since $\cosh k \neq 0$)
So $A = B = 0$; ϕ is trivial; and λ is not an eigenvalue. 2

[12] b) *(12 points)* Check whether $\lambda = 0$ is an eigenvalue. Show all work.

If $\lambda = 0$, the general solution of the ODE is
$\phi(x) = Ax + B$, 2 and $\phi'(x) = A$. 2
Then $\phi'(0) = 0 \Rightarrow A = 0$ 2 $\Rightarrow \phi(x) = B$, 2 and
$\phi(1) = 0 \Rightarrow B = 0$. 2 So $A = B = 0$; ϕ must be trivial;
and $\lambda = 0$ is not an eigenvalue. 2

[24] c) *(24 points)* Find all positive eigenvalues, and give the corresponding eigenfunctions.

If $\lambda = k^2$, $k > 0$, the general solution of the ODE is
$\phi(x) = A\cos kx + B\sin kx$, 2 and
$\phi'(x) = -Ak\sin kx + Bk\cos kx$. 2
Then $\phi'(0) = 0 \Rightarrow -Ak\cdot 0 + Bk\cdot 1 = 0$ 2 $\Rightarrow B = 0$, 2
so $\phi(x) = A\cos kx$. 2 Hence $\phi(1) = 0 \Rightarrow A\cdot\cos k = 0$. 2
To have a non-trivial solution we ~~must~~ have $A \neq 0$, so $\cos k = 0$, 2
so $k = \frac{n\pi}{2}$ $(n = 1, 3, 5, 7, \ldots)$. 2 So the eigenvalues are
$\lambda = \frac{n^2\pi^2}{4}$ $(n = 1, 3, 5, 7, \ldots)$ 4 ; with corresponding eigenfunctions $\phi(x) = \cos\left(\frac{n\pi x}{2}\right)$ 4

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2. *(28 points)* Consider the heat equation

$$\frac{\partial u}{\partial t} = 3\,\frac{\partial^2 u}{\partial x^2},$$

for $0 < x < 4$ and $t > 0$, subject to the boundary conditions

$$\frac{\partial u}{\partial x}(0,t) = 0 \quad \text{and} \quad \frac{\partial u}{\partial x}(4,t) = 0.$$

and the initial condition

$$u(x,0) = f(x) = \begin{cases} 1 & \text{for } 0 < x < 1 \\ 0 & \text{for } 1 < x < 4 \end{cases}.$$

[22] a) Give a series for the solution $u(x,t)$ of the problem. (You do not have to explain where the series comes from). Include a formula for the coefficients in the series, evaluating any integrals that appear.

The method of separation of variables gives separated solutions 2

$u(x,t) = \cos\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$, where here $L=4$ and $k=3$,

for $n = 0,1,2,3,\dots$ ~~So take~~

$$\boxed{u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{4}\right) e^{-3\left(\frac{n\pi}{4}\right)^2 t}}.$$

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As seen in class (or in the table written on the board at the test)

$$\boxed{A_0} = \frac{1}{L}\int_0^L f(\bar{x})\,d\bar{x} = \frac{1}{4}\int_0^4 f(\bar{x})\,d\bar{x} = \frac{1}{4}\int_0^1 1\,dx = \frac{1}{4}\cdot 1 = \boxed{1/4}$$ and

2 2 2

for $n = 1,2,3,\dots$,

$$\boxed{A_n} = \frac{2}{L}\int_0^L f(\bar{x})\cos\left(\frac{n\pi \bar{x}}{L}\right)d\bar{x} = \frac{2}{4}\int_0^4 f(\bar{x})\cos\left(\frac{n\pi\bar{x}}{4}\right)d\bar{x} =$$

$$= \frac{1}{2}\int_0^1 \cos\left(\frac{n\pi\bar{x}}{4}\right)d\bar{x} = \frac{1}{2}\cdot\frac{4}{n\pi}\left[\sin\left(\frac{n\pi\bar{x}}{4}\right)\right]_0^1 = \boxed{\frac{2}{n\pi}\sin\frac{n\pi}{4}}.$$

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[6] b) Write out the first three terms of the series you found in part a).

$$u(x,t) = A_0 + A_1\cos\left(\frac{\pi x}{4}\right)e^{-\frac{3\pi^2}{16}t} + A_2\cos\left(\frac{2\pi x}{4}\right)e^{-3\left(\frac{4\cdot\pi^2}{16}\right)t} + \dots$$

$$= \frac{1}{4} + \frac{2}{\pi}\sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi x}{4}\right)e^{-\frac{3\pi^2}{16}t} + \frac{2}{2\pi}\sin\left(\frac{2\pi}{4}\right)\cos\left(\frac{\pi x}{2}\right)e^{-\frac{3\pi^2}{4}t} + \dots$$

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$$= \frac{1}{4} + \frac{2}{\pi}\cdot\frac{\sqrt{2}}{2}\cos\left(\frac{\pi x}{4}\right)e^{-\frac{3\pi^2 t}{16}} + \frac{1}{\pi}\cdot 1\cdot\cos\left(\frac{\pi x}{2}\right)e^{-\frac{3\pi^2}{4}t} + \dots$$

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3. *(12 points)* Suppose $f(x)$ is a function defined for $a \le x \le b$, and $f(x) = \sum_{n=1}^{\infty} B_n \phi_n(x)$, where the functions $\phi_n(x)$ satisfy

$$\int_a^b \phi_m(x)\phi_n(x)\,dx = 7\delta_{mn}$$

for all values of m and n.
Give a formula for B_m in terms of $f(x)$, and explain how this formula is obtained.

Multiply $f(x) = \sum_{n=1}^{\infty} B_n \phi_n(x)$ by $\phi_m(x)$ (2) and integrate from $x=a$ to $x=b$ to get (2)

$$\int_a^b f(x)\,\phi_m(x)\,dx = \int_a^b \sum_{n=1}^{\infty} B_n \phi_n(x)\phi_m(x)\,dx$$

$$= \sum_{n=1}^{\infty} B_n \int_a^b \phi_n(x)\phi_m(x)\,dx \quad (2) \quad (2)$$

$$= \sum_{n=1}^{\infty} B_n \cdot 7\delta_{mn} = 7 \cdot B_m.$$

Dividing by 7 gives

$$\boxed{B_m = \frac{1}{7}\int_a^b f(x)\,\phi_m(x)\,dx.} \quad (4)$$

4. *(12 points)* For the equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial u}{\partial x} + 3\frac{\partial^2 u}{\partial x^2},$$

substitute $u(x,t) = \phi(x)G(t)$ and separate variables to find two ordinary differential equations for $\phi(x)$ and $G(t)$. You need not solve these equations.

$\frac{\partial^2 u}{\partial t^2} = \phi(x)\,G''(t)$ and $\frac{\partial u}{\partial x} = \phi'(x)G(t)$ (2) and $\frac{\partial^2 u}{\partial x^2} = \phi''(x)G(t)$ (2),

so the PDE becomes $\phi(x)G''(t) = \phi'(x)G(t) + 3\phi''(x)G(t)$.

Dividing by $\phi(x)G(t)$ gives $\frac{G''(t)}{G(t)} = \frac{\phi'(x)}{\phi(x)} + \frac{3\phi''(x)}{\phi(x)}$ (2)

Since one side of the equation is a function of t alone, and the other side is a function of x alone, they must both be constant. Call this constant λ (2). Then

$\frac{G''(t)}{G(t)} = \lambda \Rightarrow \boxed{G''(t) = \lambda G(t)}$ (2) and $\frac{\phi'(x) + 3\phi''(x)}{\phi(x)} = \lambda \Rightarrow$ (2) $\boxed{\phi'(x) + 3\phi''(x) = \lambda\phi(x)}$

Note: any other name for the constant, such as "$-\lambda$", is also okay.