EXAM 2
Math 4163
3-27-15

Name key

There are four problems totalling 100 points. You may write on the back sides of the pages if you need more room.

1. (20 points) Consider the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2},$$

for $0 < x < 1$, $t > 0$, subject to the boundary conditions

$$u(0,t) = 0 \quad \text{and} \quad u(1,t) = 0 \quad \text{for } t \geq 0$$

and the initial conditions

$$u(x,0) = 0 \quad \text{and} \quad \frac{\partial u}{\partial t}(x,0) = g(x) \quad \text{for } 0 \leq x \leq 1.$$

Give a series for the solution $u(x,t)$ of the problem. (You do not have to explain where the series comes from.) Include formulas for the coefficients in the series, in terms of the given data.

Separated solutions $u(x,t) = \phi(x)G(t)$ satisfy $\phi'' + \lambda\phi = 0$ and $G'' + c^2\lambda G = 0$, with $\phi(0) = 0$ and $\phi(1) = 0$ from the boundary conditions. So eigenvalues are $\lambda = -n^2\pi^2$ and $\phi(x) = \sin(n\pi x)$ for $n = 1, 2, 3, \ldots$, and $G(t) = \cos(cn\pi t)$ or $G(t) = \sin(cn\pi t)$. 8

Take $\boxed{u(x,t) = \sum_{n=1}^{\infty} \sin(n\pi x)\left(A_n \cos(n\pi c t) + B_n \sin(n\pi c t)\right)}$

Putting $t=0$ gives $0 = \sum_{n=1}^{\infty} A \sin(n\pi x)$, so $\boxed{A_n = 0}$ for all n, 4

and $u(x,t) = \sum_{n=1}^{\infty} B_n \sin(n\pi x)\sin(n\pi c t)$

So $u_t(x,t) = \sum_{n=1}^{\infty} n\pi c B_n \sin(n\pi x)\cos(n\pi c t)$. 2

Putting $t=0$ gives $g(x) = \sum_{n=1}^{\infty} n\pi c B_n \sin(n\pi x)$, so 2

$$\boxed{B_n = \frac{1}{n\pi c}\cdot\frac{2}{1}\int_0^1 g(x)\sin(n\pi x)\,dx}$$ 4

2. *(40 points)* Consider Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

for $0 < x < L$ and $0 < y < H$, subject to the boundary conditions

$$\frac{\partial u}{\partial x}(0, y) = 0 \quad \text{and} \quad \frac{\partial u}{\partial x}(L, y) = 0 \quad \text{for } 0 \le y \le H, \text{ and}$$

$$u(x, 0) = 0 \quad \text{and} \quad \frac{\partial u}{\partial y}(x, H) = f(x) \quad \text{for } 0 \le x \le L.$$

[20] a) Find all the separated solutions $u(x, y) = \phi(x)h(y)$ of Laplace's equation with the three homogeneous boundary conditions. (You do not have to give all the details, if they repeat results we've already done in class. But be careful to find *all* separated solutions.)

The boundary conditions imply $\phi'(0)=0$ and $\phi'(L)=0$, so

$$\phi'' + \lambda\phi = 0 \implies \begin{cases} \lambda = 0 \text{ and } \phi(x) = 1 \text{ or} \\ \lambda = \left(\frac{n\pi}{L}\right)^2 \text{ and } \phi(x) = \cos\left(\frac{n\pi x}{L}\right), \ (n = 1, 2, 3, \ldots) \end{cases} \quad 5$$

Then $h'' - \lambda h = 0$ and $h(0) = 0$, so

* if $\lambda = 0$ then $h(y) = A + By$, so $0 = A$ and $h(y) = By$ 5

* if $\lambda = \left(\frac{n\pi}{L}\right)^2$ $(n = 1, 2, 3, \ldots)$ then $h(y) = A\cosh\left(\frac{n\pi y}{L}\right) + B\sinh\left(\frac{n\pi y}{L}\right)$, and $0 = A$, so $h(y) = B\sinh\left(\frac{n\pi y}{L}\right)$. 5

Thus separated solutions are $\boxed{u(x,y) = 1 \cdot y}$ or $\boxed{u(x,y) = \cos\left(\frac{n\pi x}{L}\right)\sinh\left(\frac{n\pi y}{L}\right)}$ 5

[20] b) Write out the solution $u(x, y)$ of Laplace's equation with all four boundary conditions. Include formulas for any coefficients which appear, in terms of the given data.

Put $\boxed{u(x,y) = A_0 y + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right)\sinh\left(\frac{n\pi y}{L}\right),}$ 5

Then $\frac{\partial u}{\partial y} = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) \cdot \frac{n\pi}{L}\cosh\left(\frac{n\pi y}{L}\right)$ 5

Putting $y = H$ gives

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right)\frac{n\pi}{L}\cosh\left(\frac{n\pi H}{L}\right), \text{ and so}$$ 4

3 $\boxed{A_0 = \frac{1}{L}\int_0^L f(x)\,dx}$ and 3 $\boxed{A_n = \frac{1}{\left(\frac{n\pi}{L}\right)\cosh\left(\frac{n\pi H}{L}\right)} \cdot \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx}$

$(n = 1, 2, 3, \ldots)$

3. *(20 points)* Consider Laplace's equation for a function $u(r,\theta)$ defined outside a disc of radius 2 ($r > 2$), subject to the boundary condition

$$\frac{\partial u}{\partial r}(2,\theta) = f(\theta) \quad \text{for } -\pi \le \theta \le \pi$$

and the condition that $\lim_{r\to\infty} u(r,\theta)$ is not infinite.

[5] a) In class we found the following separated solutions of Laplace's equation in polar coordinates:

$$r^n \cos(n\theta), \quad r^n \sin(n\theta), \quad r^{-n}\cos(n\theta), \quad r^{-n}\sin(n\theta),$$

for $n = 0, 1, 2, 3, \ldots$. Which of these solutions satisfy the condition that $\lim_{r\to\infty} u(r,\theta)$ is not infinite?

When $n=0$, these solutions are either trivial or 1, and $u \equiv 1$ does satisfy the condition.
For $n = 1, 2, 3, \ldots$, only $r^{-n}\cos(n\theta)$ and $r^{-n}\sin(n\theta)$ satisfy the condition.

[15] b) Using your answer to part a), give a series for the solution $u(r,\theta)$ of the above problem. Include formulas for any coefficients which appear, in terms of the given data.

Take $\boxed{u(r,\theta) = A_0 \cdot 1 + \sum_{n=1}^{\infty} r^{-n}(A_n \cos n\theta + B_n \sin n\theta)}$ 4

Then $\frac{du}{dr} = 0 + \sum_{n=1}^{\infty} (-n) r^{-n-1}(A_n \cos n\theta + B_n \sin n\theta)$, 4

so putting $r=2$ gives

$$f(\theta) = \sum_{n=1}^{\infty} \frac{-n}{2^{n+1}}(A_n \cos n\theta + B_n \sin n\theta). \quad (-\pi \le \theta \le \pi)$$ 2

This does not give a formula for A_0 (in fact, A_0 is not determined by the boundary condition). But we do get 5

$$\boxed{A_n = \frac{2^{n+1}}{(-n)} \cdot \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos n\theta \, d\theta} \text{ and}$$

$$\boxed{B_n = \frac{2^{n+1}}{(-n)} \cdot \frac{1}{\pi}\int_{-\pi}^{\pi} g(\theta)\sin n\theta \, d\theta} \quad \text{for } n = 1, 2, 3, \ldots$$

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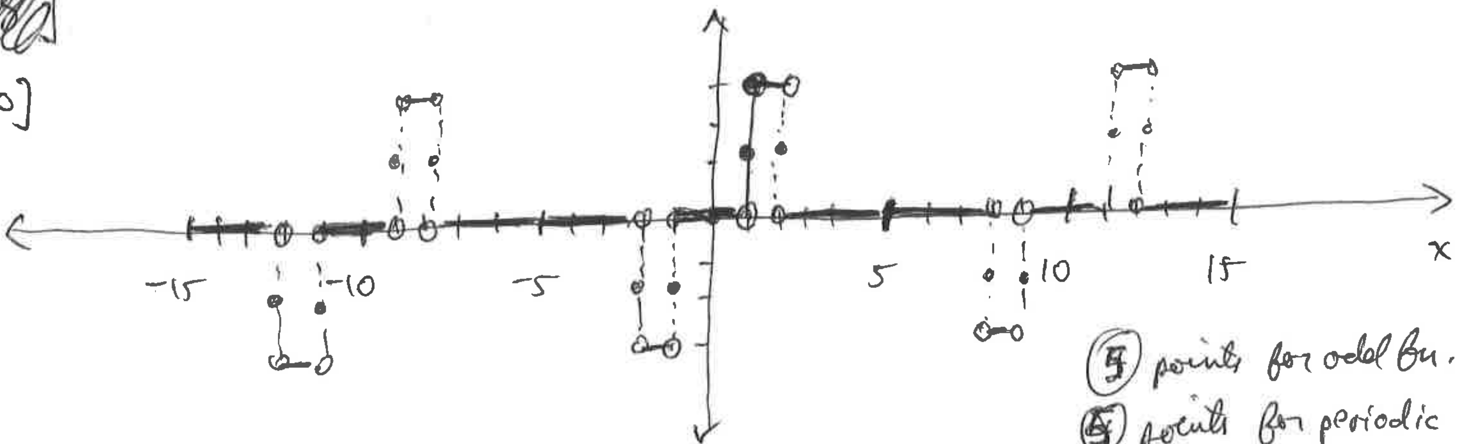
4. *(20 points)* For the function $f(x)$ defined for $0 \le x \le 5$ by

$$f(x) = \begin{cases} 0 & \text{if } 0 \le x \le 1 \\ 3 & \text{if } 1 < x < 2 \\ 0 & \text{if } 2 \le x \le 5, \end{cases}$$

[10] a) Sketch the graph of the function to which the Fourier sine series for $f(x)$ converges for $-15 \le x \le 15$.



5 points for odd fn.
5 points for periodic with period 10.

[10] b) Compute the coefficients of the Fourier sine series for $f(x)$ on $[0, 5]$. You need not evaluate the trig functions which appear in your answer.

$$B_n = \frac{2}{5}\int_0^5 f(x) \sin\left(\frac{n\pi x}{5}\right)dx \quad (3)$$

$$= \frac{2}{5}\int_1^2 3 \sin\left(\frac{n\pi x}{5}\right)dx \quad (5)$$

$$= \frac{6}{5} \; \frac{5}{n\pi}\left(-\cos\left(\frac{n\pi x}{5}\right)\right)\Bigg|_{x=1}^{x=2} \quad (2) \quad = \boxed{\frac{6}{n\pi}\left[\cos\left(\frac{n\pi}{5}\right) - \cos\left(\frac{2n\pi}{5}\right)\right] \quad n = 1, 2, 3, \ldots}$$