EXAM 3
Math 4163
4-24-15

Name key

There are three problems totalling 100 points. You may write on the back sides of the pages if you need more room.

1. *(40 points)* Consider the equation

$$\frac{\partial u}{\partial t} = 3\,\frac{\partial^2 u}{\partial x^2} + 4\,\frac{\partial^2 u}{\partial y^2}\,,$$

for $0 < x < L$, $0 < y < H$, and $t > 0$; subject to the boundary conditions

$$u(0,y,t) = 0, \quad u(L,y,t) = 0, \quad u(x,0,t) = 0, \quad u(x,H,t) = 0,$$

and the initial condition

$$u(x,y,0) = \alpha(x,y),$$

where α is a given function.

[30] **a)** Use the method of separation of variables to find all solutions of the form $u(x,y,t) = f(x)g(y)h(t)$ to the PDE and the boundary conditions.

From the PDE, we get $fgh' = 3f''gh + 4fg''h$, so

$\frac{h'(t)}{h(t)} = \frac{3f''(x)}{f(x)} + \frac{4g''(y)}{g(y)} = -\lambda$. (3) So $h'(t) = -\lambda h(t)$ (3) and

$\frac{3f''(x)}{f(x)} = \frac{-4g''(y)}{g(y)} - \lambda = -\mu$. So $\begin{cases} f''(x) = \left(-\frac{\mu}{3}\right) f(x) \text{ (3) and} \\ g''(y) = \frac{-(\lambda-\mu)}{4} g(y) \text{ (3)} \end{cases}$

Since $f(0) = f(L) = 0$, we get $\frac{\mu}{3} = \left(\frac{m\pi}{L}\right)^2$ (3) $(m = 1,2,3,\ldots)$

and $f(x) = \sin\left(\frac{m\pi x}{L}\right)$ (3). Since $g(0) = g(H) = 0$, we get

(3) $\frac{\lambda-\mu}{4} = \left(\frac{n\pi}{H}\right)^2$ $(n = 1,2,3,\ldots)$ and $g(x) = \sin\left(\frac{n\pi y}{H}\right)$ (3). Then

$\mu = 3\left(\frac{m\pi}{L}\right)^2$ and $\lambda_{mn} = 4\left(\frac{n\pi}{H}\right)^2 + \mu = 4\left(\frac{n\pi}{H}\right)^2 + 3\left(\frac{m\pi}{L}\right)^2$ (3),

and $h(t) = e^{-\lambda t}$ (3) So $\boxed{u(x,y,t) = \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi y}{H}\right)e^{-\lambda_{mn}t}}$

b) Find a linear combination of the solutions from part a) which satisfies the given initial condition. Express the coefficients which appear in terms of the function α.

[10] $u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi y}{L}\right)e^{-\lambda_{mn}t}$ (4)

and $A_{mn} = \frac{\int_0^H\int_0^L \alpha(x,y)\sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi y}{H}\right)dx\,dy}{\int_0^H\int_0^L \sin^2\left(\frac{m\pi x}{L}\right)\sin^2\left(\frac{n\pi y}{H}\right)dx\,dy}$ (4) $= \boxed{\frac{4}{LH}\int_0^H\int_0^L \alpha \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi y}{H}\right)dx\,dy}$ (2)

2. (30 points) Consider the boundary-value problem

$$\phi''(x) + \lambda\phi(x) = 0 \quad \text{for } 0 < x < 5$$
$$\phi'(0) = 2\phi(0)$$
$$\phi'(5) = -3\phi(5).$$

a. What is the general solution of the ODE for $\lambda = 0$?

[4]

$\phi(x) = Ax + B$

[18]

b. Decide whether $\lambda = 0$ is an eigenvalue for this problem. Give a complete reason for your answer.

We have $\phi'(x) = A$ (3), so $\phi(0) = B$ and $\phi'(0) = A$, and

$\phi'(0) = 2\phi(0) \Rightarrow A = 2B$. (3)

also $\phi(5) = 5A + B$ and $\phi'(5) = A$, (3) so

$\phi'(5) = -3\phi(5) \Rightarrow A = -3(5A+B) \Rightarrow A = -15A - 3B$

$\Rightarrow$ $16A = -3B$. (3) Since $A = 2B$, this gives $32B = -3B$,

so $35B = 0$, so $B = 0$ (3), and hence $A = 2B = 0$.

Thus the boundary conditions force $A = B = 0$, so $\lambda = 0$ is not an eigenvalue. (3)

c. Use the Rayleigh quotient

[8]

$$\lambda = \frac{\phi(0)\phi'(0) - \phi(L)\phi'(L) + \int_0^L (\phi'(x))^2\, dx}{\int_0^L (\phi(x))^2\, dx}.$$

to show that this problem has no negative eigenvalues.

Since $\phi'(0) = 2\phi(0)$ and $\phi'(5) = -3\phi(5)$, Then

The Rayleigh quotient gives, for this problem (where $L = 5$)

$$\lambda = \frac{\phi(0)\cdot 2\phi(0) - \phi(5)[-3\phi(5)] + \int_0^L \phi'(x)^2\, dx}{\int_0^L \phi(x)^2\, dx} \quad (4)$$

or

$$\lambda = \frac{2\phi(0)^2 + 3\phi(5)^2 + \int_0^L \phi'(x)^2\, dx}{\int_0^L \phi(x)^2\, dx}$$

Since every term in this expression is a square times a positive number, all terms are nonnegative, so $\lambda \geq 0$. (4)

That is, every eigenvalue is non-negative.

3. (30 points) Consider the eigenvalue problem

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} + \lambda \phi = 0$$

on the domain $0 < r < 1$, $0 < \theta < \frac{\pi}{6}$ (see diagram), with boundary conditions

$$\phi(1,\theta) = 0 \quad \text{and} \quad \lim_{r\to 0} \phi(r,\theta) < \infty \text{ for } 0 \le \theta \le \frac{\pi}{6},$$

and

$$\frac{\partial \phi}{\partial \theta}(r,0) = 0 \quad \text{and} \quad \frac{\partial \phi}{\partial \theta}\left(r, \frac{\pi}{6}\right) = 0 \quad \text{for } 0 \le r \le 1.$$

Find the eigenvalues and eigenfunctions by separating variables, taking $\phi(r,\theta) = f(r)g(\theta)$.
(Hint: You may assume that the equation for f is

$$f''(r) + \frac{1}{r}f'(r) + \left(\lambda - \frac{\mu}{r^2}\right) f(r) = 0,$$

with $\lambda > 0$, which transforms via $z = \sqrt{\lambda}\, r$ into Bessel's equation

$$f''(z) + \frac{1}{z}f'(z) + \left(1 - \frac{\mu}{z^2}\right) f(z) = 0.)$$

Eigenvalues: $\lambda = (z_{6m,n})^2$ 3

Eigenfunctions: $J(z_{6m,n} r) \cdot \cancel{\cos}(6m\theta)$ 3

(for $m =$ 0, 1, 2, 3, … and $n =$ 1, 2, 3, …) 3

We get $\dfrac{f'' + \frac{1}{r} f'}{f} + \dfrac{1}{r^2}\dfrac{g''}{g} + \lambda = 0$, or $\dfrac{r^2 f'' + r f' + r^2 \lambda f}{f} = \dfrac{-g''}{g} = \mu$,

so $g''(\theta) = -\mu g(\theta)$ 3 and $r^2 f'' + r f' + r^2 \lambda f - \mu f = 0$, or

$f''(r) + \frac{1}{r} f'(r) + \left(\lambda - \frac{\mu}{r^2}\right) f(r) = 0$. From the boundary conditions we get $g'(0) = g'\left(\frac{\pi}{6}\right) = 0$, so $\mu = \left(\frac{m\pi}{\pi/6}\right)^2 = (6m)^2$ 3, $g(\theta) = \cancel{\cos}(6m\theta)$ 3 $(m = 0, 1, 2, 3, \dots)$. From the hint we get that for $z = \sqrt{\lambda} r$, $f(z)$ solves $f''(z) + \frac{1}{z} f'(z) + \left(1 - \frac{(6m)^2}{z^2}\right) f(z) = 0$

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so $f(z) = A J_{6m}(z) + B Y_{6m}(z)$, or $f(r) = A J_{6m}(\sqrt{\lambda} r) + B Y_{6m}(\sqrt{\lambda} r)$.

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From $\lim_{r\to 0} f(r) = 0$ we get $B = 0$, and from $f(1) = 0$ we get (for non-trivial solutions) $J_{6m}(\sqrt{\lambda}) = 0$. So $\sqrt{\lambda} = z_{6m,n}$ 3, a zero of $J_{6m}(z)$ $(n = 1, 2, 3, \dots)$ 3