Solutions to problems on Exam 3

[12] (*) 3. We are given that $\lim\limits_{x\to 3}\left[\frac{f(x)-7}{x-3}\right]$ exists and equals 4.

We also know that $\lim\limits_{x\to 3}(x-3)$ exists and equals 0.

So by the theorem (2) on products of limits (4.2.4),

$$\lim_{x\to 3}\,(x-3)\left[\frac{f(x)-7}{x-3}\right] \text{ exists and equals } 4\cdot 0 = 0. \quad (2)$$

But $(x-3)\left[\frac{f(x)-7}{x-3}\right] = f(x)-7$ (for $x\neq 3$), so it follows that

$$\lim_{x\to 3}\big(f(x)-7\big) \text{ exists and equals } 0. \quad (2)$$

Also, we know that $\lim\limits_{x\to 3} 7$ exists and equals 7.

So by the theorem (2) on sums of limits (4.2.4),

$$\lim_{x\to 3}\left[7+\big(f(x)-7\big)\right] \text{ exists and equals } 7+0=7. \quad (2)$$

Hence $\lim\limits_{x\to 3} f(x)$ exists and equals 7. (2)

[12] 4. We have $\lim\limits_{x\to 3} g(x) = \lim\limits_{x\to 3}\left(\frac{x^2-9}{x-3}\right)$ (2) $= \lim\limits_{x\to 3}\frac{(x-3)(x+3)}{x-3}$ (2) $=$

$= \lim\limits_{x\to 3}(x+3) = 3+3 = 6$ (2). So $\lim\limits_{x\to 3} g(x) = 6$. Also we are

given that $g(3) = A$ (2). Since g is continuous at 3, then

$\lim\limits_{x\to 3} g(x) = g(3)$ (2). So $6 = A$. (2)

(*) For an alternate proof of 3., see the last page.

[10]

5. a. For all $x \neq 0$ we have $\frac{f(x)-f(0)}{x-0}$ (2) $= \frac{(|x|+1)-(|0|+1)}{x-0}$ (2) $=$

$= \frac{(|x|+1)-1}{x} = \frac{|x|}{x} = \begin{cases} 1 & \text{if } x>0 \\ -1 & \text{if } x<0 \end{cases}$ (2). We proved

in class that the function $\begin{cases} 1 & \text{if } x>0 \\ -1 & \text{if } x<0 \end{cases}$ does not have

a limit at 0 (2). So $\frac{f(x)-f(0)}{x-0}$ does not have a limit

at 0 (2). This proves f is not differentiable at 0.

[10] b. For all $x \neq 0$ we have $\frac{g(x)-g(0)}{x-0}$ (2) $= \frac{(|x|+1)x-(|0|+1)\cdot 0}{x-0} =$

$= \frac{(|x|+1)x-0}{x} = |x|+1$ (2). That is,

$$\frac{g(x)-g(0)}{x-0} = |x|+1 \quad (\text{for all } x \neq 0).$$

We know that $\lim_{x\to 0} |x| = 0$ (2) (you could easily prove this using the sequential criterion), ~~or you could use the squeeze theorem~~

so it follows from the theorem on the sum of limits (4.2.4) that

$$\lim_{x\to 0} \frac{g(x)-g(0)}{x-0} = \lim_{x\to 0}|x| + \lim_{x\to 0} 1 = 0 + 1 = 1 \text{ (2)}.$$

This proves that g is differentiable at 0, and $g'(0)=1$. (2)

[12] 6. Since f and g are continuous on $[0,1]$, it follows from the theorem (2) on sums and products of continuous functions (5.2.1) that $g-f$ (2) is continuous on $[0,1]$. We are given that for all $x > 0$, $f(x) < g(x)$, so $g(x)-f(x) > 0$, so $(g-f)(x) > 0$. (2) Then from a homework problem done in class, (2) there exists $\alpha > 0$ such that for all $x \in [0,1]$, $(g-f)(x) \geq \alpha$. (2) Hence for all $x \in [0,1]$, $g(x) - f(x) \geq \alpha$, so $g(x) \geq f(x) + \alpha$. (2)

Alternate proof of 3. Let $g(x) = \begin{cases} f(x) & \text{if } x \neq 3 \\ 7 & \text{if } x = 3 \end{cases}$. (2)

Then $\lim_{x\to 3} \frac{g(x)-g(3)}{x-3} = \lim_{x\to 3} \frac{f(x)-7}{x-3} = 4$, (2) so $g'(3)$ exists and equals 4. (2) Since g is differentiable at 3, then g is continuous at 3. (2) So $\lim_{x\to 3} g(x) = g(3) = 7$. (2)

But $\lim_{x\to 3} f(x) = \lim_{x\to 3} g(x)$ (since $f(x) = g(x)$ for all $x \neq 3$), so $\lim_{x\to 3} f(x) = 7$. (2)