(4.) (a) First we claim that $\sup T$ is an upper bound for S. To see this, let $x \in S$ be given. Then $|x| \in T$ by definition of T. Since $\sup T$ is an upper bound of T, it follows that $|x| \leq \sup T$. Since $x \leq |x|$ for every $x \in \mathbb{R}$, it follows that $x \leq \sup T$. This shows $\sup T$ is an upper bound for S.

Now since $\sup T$ is an upper bound of S and $\sup S$ is by definition the least upper bound of S, it follows that $\sup S \leq \sup T$.

(b.) Take $S = \{-1, 0\}$. Then $T = \{0, 1\}$, and we have $\sup S = 0$ and $\sup T = 1$, so $\sup S \neq \sup T$.

(5.) We have $\dfrac{3n^2+n}{5n^2-6} = \dfrac{3+\frac{1}{n}}{5-6/n^2}$. Since $\lim\left(\frac{1}{n}\right) = 0$ (proved in class) and $0 \leq \frac{1}{n^2} \leq \frac{1}{n}$, then $\lim\left(\frac{1}{n^2}\right) = 0$. Therefore by the Theorem on sums and products of convergent sequences, $\lim\left(\frac{-6}{n^2}\right) = (-6)\lim\left(\frac{1}{n^2}\right) = (-6)\cdot 0 = 0$ and $\lim\left(5 - \frac{6}{n^2}\right) = 5 - 0 = 5$.

$\boxed{\text{Also, } \lim\left(3+\frac{1}{n}\right) = \lim(3) + \lim\left(\frac{1}{n}\right) = 3+0 = 3}$

Since $\lim\left(5 - \frac{6}{n^2}\right) \neq 0$, it follows from the Theorem on quotients of convergent sequences that $\lim\left(\frac{3+\frac{1}{n}}{5-6/n^2}\right) = \frac{5}{3}$.

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(6) (a)
* If $n = 1$, then $4^n = 4$ and $n^2 = 1$, so $4^n \geq n^2$.
* Suppose $4^n \geq n^2$. Then $4^{n+1} = 4 \cdot 4^n \geq 4n^2$.

Also $(n+1)^2 = n^2 + 2n + 1 \leq n^2 + 2n^2 + n^2 = 4n^2$.

So $(n+1)^2 \leq 4n^2 \leq 4^{n+1}$, so $(n+1)^2 \leq 4^{n+1}$.

This completes the proof by induction.

6. b.

Since $4^n \geq n \cdot n$ for all $n \in \mathbb{N}$,

Then $1 \geq \frac{n \cdot n}{4^n}$, so $\frac{1}{n} \geq \frac{n}{4^n}$ for all $n \in \mathbb{N}$.

Hence $\left|\frac{n}{4^n} - 0\right| = \left|\frac{n}{4^n}\right| = \frac{n}{4^n} \leq \frac{1}{n}$.

Since $\lim\left(\frac{1}{n}\right) = 0$, it follows from a Theorem proved in class that $\lim\left(\frac{n}{4^n}\right) = 0$

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7. Let $\varepsilon > 0$ be given.

Since we are given that $\lim(x_n) = 0$, then by definition of limit we can find a natural number K such that if $n \geq K$ then

$$|x_n - 0| < \frac{\varepsilon}{2}.$$

Hence $|x_n| < \frac{\varepsilon}{2}$ for $n \geq K$.

Since $|y_n| \leq 2$ for all $n \in \mathbb{N}$, it follows that

$$|x_n y_n| \leq 2 \cdot \frac{\varepsilon}{2} = \varepsilon \quad \text{for } n \geq K.$$

Hence

$$|x_n y_n - 0| \leq \varepsilon \quad \text{for } n \geq K.$$

This proves $\lim(x_n y_n) = 0$.