Introduction to Analysis
Exam 2

1. *(5 points)* State the Bolzano-Weierstrass theorem.

2. *(5 points)* Define "Cauchy sequence". What is the relationship between Cauchy sequences and convergent sequences?

3. *(5 points)* Give the definition of limit of a function at a point.

4. *(25 points)* Let (x_n) be defined recursively by: $x_1 = 1$, and $x_{n+1} = (x_n + 3)/8$ for all $n \in \mathbf{N}$.

a. Prove that (x_n) converges.
b. Find its limit.

5. *(10 points)* Suppose $\lim(x_n) = 3$ and $\lim(y_n) = 4$. Let (z_n) be defined by

$$z_n = \begin{cases} x_k & \text{if } n = 2k \\ y_k & \text{if } n = 2k+1. \end{cases}$$

Show that (z_n) diverges.

6. *(15 points)* Suppose that (x_n) is an increasing sequence. Prove that either (x_n) converges, or $\lim(x_n) = +\infty$.

7. *(15 points)* Let (s_n) be the sequence defined by $s_n = \sum_{k=1}^{n} \frac{1}{k!}$.

a. Prove that (s_n) is bounded above. (Hint: can you find a larger sequence that you know to be bounded?)

b. Deduce from part **a** that (s_n) converges.

8. *(20 points)* Let $f(x)$ be defined for all $x \in \mathbf{R}$ by $f(x) = 5x + 7$.

a. Prove that $\lim_{x \to 3} f(x) = 22$ using the definition of limit of a function at a point.

b. Prove that $\lim_{x \to 3} f(x) = 22$ using the sequential criterion for limits.

(4.) (a) First we show that (x_n) is decreasing; that is, $x_{n+1} \le x_n$ for all $n \in \mathbb{N}$.
Since $x_1 = 1$ and $x_2 = (1+3)/8 = \frac{1}{2}$, the statement $x_2 \le x_1$ is true.
Now suppose $x_{n+1} \le x_n$. Then $x_{n+1} + 3 \le x_n + 3$, so $\frac{x_{n+1}+3}{8} \le \frac{x_n+3}{8}$,
(5) so $x_{n+2} \le x_{n+1}$. This proves that (x_n) is decreasing by induction.

Next we use induction to show that $x_n \ge 0$ for all $n \in \mathbb{N}$.
(5) We have $x_1 = 1 \ge 0$, and if $x_n \ge 0$ then $(x_n + 3)/8 \ge \frac{3}{8} > 0$,
so $x_{n+1} \ge 0$, This completes the induction.

We have now shown that (x_n) is decreasing and
(5) bounded below (by 0), so by the Monotone Convergence Theorem, (x_n) converges.

(b) From (a) we know that $x = \lim(x_n)$ exists. Also from
(5) class we know that if $\lim(x_n) = x$ then $\lim(x_{n+1}) = x$ as well.
Therefore, from Theorem 3.2.3 (on the limits of sums and products
(5) of sequences) we have $x = \lim(x_{n+1}) = \lim\left(\frac{x_n+3}{8}\right) =$
$= \frac{\lim(x_n)+3}{8} = \frac{x+3}{8}$. Solving $x = \frac{x+3}{8}$ gives $7x = 3$ or $x = \frac{3}{7}$.

(5.) From the definition of (z_n) we see immediately that (z_n)
(10) has two subsequences which converge to different limits:
namely $\lim(z_{2k}) = \lim(x_k) = 3$ and $\lim(z_{2k+1}) = \lim(y_k) = 4$.
But If (z_n) converged, then all its subsequences would converge to the same limit. So (z_n) must not converge.

(6.) There are two possibilities: either (x_n) is bounded, or it is not bounded. We will show that (x_n) converges in the first case and $\lim(x_n) = +\infty$ in the second case.

(5) If (x_n) is bounded, then since (x_n) is increasing, it converges, by the Monotone Convergence Theorem.

If (x_n) is not bounded, then let $M > 0$ be given.

(cont'd) $\longrightarrow$

6. (cont'd).

5 Since M is not an upper bound for (x_n), there must exist some $K \in \mathbb{N}$ such that $x_K \geq M$. But since (x_n) is

5 increasing, it follows that for every $n \geq K$, $x_n \geq x_K$, and hence $x_n \geq M$ also. This shows that $\lim(x_n) = +\infty$.

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7. (a) One way to do this problem is to first look at the sequence (t_n) defined by

$$t_n = \sum_{k=1}^{n} \frac{1}{2^{k-1}}.$$

5 Clearly, for all $k \in \mathbb{N}$,

$$2^{k-1} = 1 \cdot \underbrace{2 \cdot 2 \cdot 2 \cdots 2}_{k-1 \text{ times}} \leq 1 \cdot 2 \cdot 3 \cdot 4 \cdots k = k!,$$

since each factor in the product for 2^{k-1} is less than the corresponding factor in the product for $k!$. (This can also be proved more rigorously by induction.) Since $2^{k-1} \leq k!$, Then $\frac{1}{k!} \leq \frac{1}{2^{k-1}}$, so $\sum_{k=1}^{n} \frac{1}{k!} \leq \sum_{k=1}^{n} \frac{1}{2^{k-1}}$, so $s_n \leq t_n$.

5 But, as shown in class, $\sum_{k=1}^{n} \frac{1}{2^{k-1}}$ converges (it is a geometric series with sum $\frac{1}{1-\frac{1}{2}} = 2$). So (t_n) converges, so t_n is bounded above (by 2). Hence (s_n) is bounded above by 2 also.

(b) Obviously (s_n) is increasing, as s_{n+1} differs from

5 s_n by the addition of a positive term: $s_{n+1} = s_n + \frac{1}{(n+1)!} \geq s_n$.

Since (s_n) is increasing, and bounded above [by (a)], Then (s_n) converges by the Monotone Convergence Theorem.

[Note: $\lim(s_n)$ turns out to equal $e-1$, as can be seen from the Taylor series for e^x with $x=1$.]

8. (a) Let $\varepsilon > 0$ be given. Choose $\delta = \frac{\varepsilon}{5}$. If $0 < |x-3| < \delta$,

[10] Then $|x-3| < \frac{\varepsilon}{5}$, so $5|x-3| < \varepsilon$, so $|5x-15| < \varepsilon$,

so $|(5x+7)-22| < \varepsilon$. This proves $\lim_{x \to 3} 5x+7 = 22$.

(b.) Let (x_n) be a sequence such that $x_n \neq 3$ for all $n \in \mathbb{N}$

[10] and $\lim(x_n) = 3$. Then by Theorem 3.2.3 on sums and products of sequences, $\lim(5x_n + 7) = 5\lim(x_n) + 7 =$ $= 5 \cdot 3 + 7 = 22$. This proves $\lim_{x \to 3}(5x+7) = 22$, by the sequential criterion for limits.