Introduction to Analysis
Exam 3

1. *(5 points)* Give the definition of a continuous function.

2. *(10 points)* Prove that the composition of continuous functions is continuous.

3. *(15 points)* Let $f(x)$ be defined by $f(x) = \frac{|x|}{x}$ for all $x \neq 0$. Either find $\lim_{x \to 0} f(x)$, or prove that it does not exist.

4. *(15 points)* Suppose $f : [0,1] \to \mathbf{R}$ is continuous on $[0,1]$, and $f(x) > 0$ for all $x \in [0,1]$. Prove that if $g(x) = 1/f(x)$, then $g(x)$ is bounded on $[0,1]$.

5. *(15 points)* Prove that there exists a real number x such that $1 + 3x - x^4 = 0$.

6. *(20 points)*

a. Give the definition of derivative.

b. If $f : \mathbf{R} \to \mathbf{R}$ is defined by

$$f(x) = \begin{cases} x & \text{if } x \in \mathbf{Q} \\ 0 & \text{if } x \notin \mathbf{Q}, \end{cases}$$

show that f is not differentiable at 0.

7. *(20 points)*

a. State the mean value theorem.

b. Use the mean value theorem to prove that $\sqrt{x} \leq \frac{1}{2}(x+1)$ for all $x > 1$.

Answers to Exam 3

(1) See definition 5.1.1 in the text

(2) See Theorem 5.2.6 in the text

(3) Let $(x_n) = (\frac{1}{n})$ and let $(y_n) = (\frac{-1}{n})$. Then $\lim(x_n) = 0$, $\lim(y_n) = 0$, and for all $n \in \mathbb{N}$, $x_n \neq 0$ and $y_n \neq 0$. Since $x_n > 0$ then

$f(x_n) = \frac{|x_n|}{x_n} = \frac{x_n}{x_n} = 1$ for all $n \in \mathbb{N}$, and since $y_n < 0$ Then

$f(y_n) = \frac{|y_n|}{y_n} = \frac{-y_n}{y_n} = -1$ for all $n \in \mathbb{N}$. So $\lim(f(x_n)) = 1$ and $\lim(f(y_n)) = -1$.

It follows from the sequential criterion for limits that $\lim\limits_{x \to 0} f(x)$ does not exist.

(4) Since $f(x) \neq 0$ for all $x \in [0,1]$ and f is continuous on $[0,1]$, it follows from a theorem proved in class that $\frac{1}{f}$ is continuous on $[0,1]$. Also, we proved in class that if a function is continuous on a closed interval then it is bounded on that interval. So $\frac{1}{f}$ is bounded on $[0,1]$.

(5) Let $f(x) = 1 + 3x - x^4$. Then $f(0) = 1 > 0$ and $f(2) = 1 + 3 \cdot 2 - 2^4 = -9 < 0$. So by the intermediate value Theorem, there exists $c \in [0,2]$ such that $f(c) = 0$.

(6.) (a) See definition 6.1.1 in the text

(b) Let $g(x) = \frac{f(x) - 0}{x - 0}$ for $x \neq 0$. If $x \in \mathbb{Q}$ Then

$g(x) = \frac{x - 0}{x - 0} = 1$ and if $x \notin \mathbb{Q}$ Then $g(x) = \frac{0 - 0}{x - 0} = 0$.

So $g(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{Q} \end{cases}$. Let (r_n) be a sequence of rational numbers such that $r_n \neq 0$ and $\lim(r_n) = 0$, and let (s_n) be a sequence of irrational numbers such that

$s_n \neq 0$ and $\lim(s_n) = 0$. Then $\lim(g(r_n)) = \lim(1) = 1$ and $\lim(g(s_n)) = \lim(0) = 0$, so it follows from the sequential criterion for limits that $\lim_{x\to 0} g(x)$ does not exist. Therefore f is not differentiable at 0.

7 5 (a) See Theorem 6.2.4 in the text

(b) Let $f(x) = \frac{1}{2}(x+1) - \sqrt{x}$, and apply the mean value Theorem to f on the interval $[1, x]$ (where $x > 1$ is given). We get that there exists $c \in (1, x)$ such that

$$f'(c) = \frac{f(x) - f(1)}{x - 1} \qquad 5$$

Now $f'(x) = \frac{1}{2}\cdot 1 - \frac{1}{2}x^{-1/2}$, so the preceding equation says that

$$\frac{1}{2} - \frac{1}{2\sqrt{c}} = \frac{f(x) - f(1)}{x-1}. \qquad 3$$

Also $f(1) = \frac{1}{2}(1+1) - \sqrt{1} = 0$, so we have

$$\frac{1}{2}\left(1 - \frac{1}{\sqrt{c}}\right)(x-1) = f(x) - 0 = f(x). \qquad 2$$

But since $c \in (1, x)$, then $c > 1$, so $\sqrt{c} > 1$, so $1 > \frac{1}{\sqrt{c}}$, 3

so $1 - \frac{1}{\sqrt{c}} > 0$. Also $x - 1 > 0$ since $x > 1$. So we can conclude that $f(x) > 0$. Therefore

$$\frac{1}{2}(x+1) - \sqrt{x} > 0,$$

so

$$\frac{1}{2}(x+1) > \sqrt{x}. \qquad 2$$