Solutions to problems on Assignment 16

(Note: you don't necessarily have to include as much detail in your arguments as in the ones given below — some of the details can be safely omitted on the grounds that they're obvious. However, what's obvious to one person is not always obvious to another.)

6.2.6. Define the function f by $f(x) := \sin x$ for all $x \in \mathbf{R}$. Now let x and y be given (fixed but arbitrary) real numbers.

If $x < y$, we can apply the Mean Value Theorem to f on the interval $[x, y]$. This gives the existence of a number $c \in (x, y)$ such that

$$f'(c) = \frac{f(y) - f(x)}{y - x}.$$

Since $f'(z) = \cos z$ for all $z \in \mathbf{R}$, and $|\cos z| \leq 1$ for all $z \in \mathbf{R}$, it follows that $|f'(c)| \leq 1$. Therefore, from the above equation we obtain that

$$\left| \frac{f(y) - f(x)}{y - x} \right| \leq 1.$$

Hence

$$\left| \frac{\sin(y) - \sin(x)}{y - x} \right| \leq 1,$$

or

$$\frac{|\sin(y) - \sin(x)|}{|y - x|} \leq 1.$$

After multiplying by $|y - x|$, this gives

$$|\sin(y) - \sin(x)| \leq |y - x|.$$

Since, for all real numbers a and b, we have $|a - b| = |b - a|$, this is the same as the desired inequality

$$|\sin(x) - \sin(y)| \leq |x - y|.$$

If $y < x$, we apply the same argument on the interval $[y, x]$, and obtain the same result.

If $y = x$, the desired inequality is obvious, because it just says that $0 \leq 0$.

So the inequality is proved in all cases.

6.2.7. Define the function f by $f(x) := \ln x$ for all $x > 0$. Now let x be a given (fixed but arbitrary) real number such that $x > 1$.

Apply the Mean Value Theorem to f on the interval $[1, x]$. This gives the existence of a number $c \in (1, x)$ such that

$$f'(c) = \frac{f(x) - f(1)}{x - 1}.$$

Now $f'(z) = 1/z$ for all $z > 0$, so $f'(c) = 1/c$. Also $f(1) = \ln 1 = 0$. So the above equation becomes

$$\frac{1}{c} = \frac{\ln x - 0}{x - 1} = \frac{\ln x}{x - 1}.$$

Since $c \in (1, x)$, then $1 < c < x$, so

$$\frac{1}{x} < \frac{1}{c} < 1,$$

and, combined with the equation in the preceding paragraph, this gives

$$\frac{1}{x} < \frac{\ln x}{x - 1} < 1.$$

Since $x > 1$, then $x - 1 > 0$, and we can multiply these inequalities by $x - 1$ without changing their direction. Hence we get

$$\frac{x - 1}{x} < \ln x < x - 1,$$

as desired.