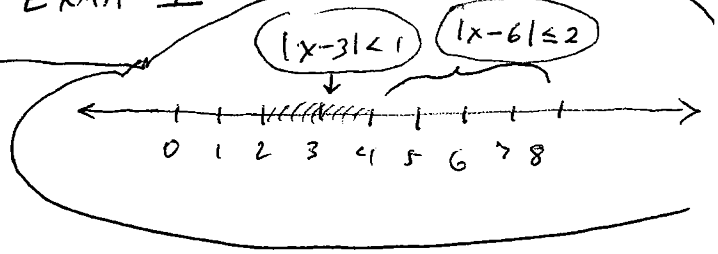
Answers to Exam 1



1 (a) If $|x-3|<1$,
then $-1 < x-3 < 1$, so
$2 < x < 4$.
Since $x<4$, then $x-6 < 4-6$, so $x-6 < -2$, so $|x-6|>2$.

(b.) As seen in (a) if $|x-3|<1$ then $2<x<4$ and $|x-6|>2$. //
Since $2<x<4$ then $4<x^2<16$ so $5<x^2+1<17$,
so $|x^2+1|<17$. Therefore $\left|\frac{x^2+1}{x-6}\right| = \frac{|x^2+1|}{|x-6|} < 17 \cdot \frac{1}{|x-6|}$.
But since $|x-6|>2$ then $\frac{1}{|x-6|} < \frac{1}{2}$, so $\left|\frac{x^2+1}{x-6}\right| < 17 \cdot \frac{1}{2}$. //

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2 Since sup S is the least upper bound of S and inf T is less than sup S, then inf T is not an upper bound of S (4). So there exists some element $x \in S$ such that inf $T < x$ (3). But then, since inf T is the greatest lower bound of T, x cannot be a lower bound of T. (4) So there exists some element $y \in T$ such that $y < x$. (4) //

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3 To prove this, it is enough to show that if $x>1$, then there exists some $n \in \mathbb{N}$ such that $x \notin I_n$.

If $x>1$, then $x-1>0$, so by the corollary to the Archimedean Property of $\mathbb{R}$, there exists $n \in \mathbb{N}$ such that $x-1 > \frac{1}{n}$. But then $x > 1+\frac{1}{n}$, so $x \notin (0, 1+\frac{1}{n})$,
so $x \notin I_n$. //

4 (a) First, use algebra to rewrite x_n as

$$x_n = \frac{n^2 \cdot n}{(n+1)\cdot n} - \frac{n^3 - 3n^2 - 2}{n^2+n} = \frac{n^3-(n^3-3n^2-2)}{n^2+n} = \frac{3n^2+2}{n^2+n} = \frac{3+\frac{2}{n^2}}{1+\frac{1}{n}}.$$

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We know from class that $\lim \frac{1}{n} = 0$ 2 and so, by the Theorem on products of limits, $\lim \frac{1}{n^2} = \lim \frac{1}{n} \cdot \lim \frac{1}{n} = 0 \cdot 0 = 0.$ 2

Since $\lim 2 = 2$ it again follows from the Theorem on products of limits that $\lim \frac{2}{n^2} = \lim 2 \cdot \lim \frac{1}{n^2} = 2 \cdot 0 = 0.$ 2

By the Theorem on sums of limits, we now have

$\lim \left(3 + \frac{2}{n^2}\right) = \lim 3 + \lim \frac{2}{n^2} = 3 + 0 = 3$ and 2

$\lim \left(1 + \frac{1}{n}\right) = \lim 1 + \lim \frac{1}{n} = 1 + 0 = 1.$

Finally, by the Theorem on quotients of limits, since the sequence $1+\frac{1}{n}$ in the denominator has the limit 1 (not 0),

Then $\lim \left(\frac{3+\frac{2}{n^2}}{1+\frac{1}{n}} \right) = \frac{\lim\left(3+\frac{2}{n^2}\right)}{\lim\left(1+\frac{1}{n}\right)} = \frac{3}{1} = 3.$ 2 So $\lim x_n = 3.$ //

[10] (b) Let $(b_n) = \left(\frac{1}{\sqrt{n}}\right)$. From class we know that $\lim \frac{1}{n} = 0$, and from a Theorem in class on square roots of limits it follows that $\lim \frac{1}{\sqrt{n}} = \lim \sqrt{\frac{1}{n}} = \sqrt{\lim \frac{1}{n}} = \sqrt{0} = 0.$

So $\lim b_n = 0.$ 4

Now for every $n \in \mathbb{N}$, we have $|x_n| = \left|\frac{(-1)^n}{\sqrt{n}}\right| = \frac{1}{\sqrt{n}} = b_n$, so in particular $|x_n| \le b_n$. 4 Then from a Theorem in class we can conclude that $\lim x_n = 0.$ 2

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5. For each $n \in \mathbb{N}$, let $k = n^2$ and $a = \frac{2}{n}$. Since $a > -1$ and $k \in \mathbb{N}$, we have by Bernoulli's inequality that

$$(1+a)^k \ge 1 + ka, \text{ or } \left(1+\frac{2}{n}\right)^{n^2} \ge 1 + n^2 \cdot \frac{2}{n}. \text{ So}$$

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$$\left(1+\frac{2}{n}\right)^{n^2} \geq 1+2n,$$

so $$\left(1+\frac{2}{n}\right)^{n^2} \geq n.$$

Now, from the Archimedean property of $\mathbb{R}$, we know that $\mathbb{N}$ is not bounded above. Combined with the preceding inequality, this tells us that the sequence $\left(1+\frac{2}{n}\right)^{n^2}$ is not bounded. 5 (For every $M \in \mathbb{R}$, there exists some $n \in \mathbb{N}$ such that $n > M$, so $\left(1+\frac{2}{n}\right)^{n^2} > M$ also.)

Since $\left(1+\frac{2}{n}\right)^{n^2}$ is not bounded, then by a theorem from class it cannot converge. // 5