# Introduction to Analysis
# Exam 3

You may use any result from class, but when you do so please be clear as to which result you are using.

**1.** *(15 points)* Let $f$ and $g$ be functions defined on $\mathbf{R}$, and suppose that there exists a number $M$ such that $|f(x)| \leq M$ for all $x \in \mathbf{R}$, and that $\lim_{x \to c} g(x) = 0$. Prove that $\lim_{x \to c} f(x) \cdot g(x) = 0$.

**2.** *(15 points)* Suppose $f$ is a function defined on $\mathbf{R}$, and let $g$ denote the function defined on $\mathbf{R}$ by $g(x) := f(2x)$. If $\lim_{x \to 0} f(x)$ exists, show that $\lim_{x \to 0} g(x) = \lim_{x \to 0} f(x)$.

**3.** *(15 points)* Let $f : \mathbf{R} \to \mathbf{R}$ be defined by

$$f(x) := \begin{cases} x^2 & \text{for } x \neq 3 \\ 7 & \text{for } x = 3. \end{cases}$$

**a.** Show that $f$ is not continuous at 3.

**b.** Show that $f$ is not differentiable at 3.

**4.** *(25 points)* Let $f : \mathbf{R} \to \mathbf{R}$ be defined by

$$f(x) := \begin{cases} x & \text{for } x \text{ rational} \\ -x & \text{for } x \text{ irrational}. \end{cases}$$

**a.** Show that $f$ is continuous at 0.

**b.** Show that $f$ is not differentiable at 0.

**5.** *(15 points)* Let $f : \mathbf{R} \to \mathbf{R}$ be defined by

$$f(x) := \begin{cases} x^2 \sin(1/x) & \text{for } x \neq 0 \\ 0 & \text{for } x = 0. \end{cases}$$

Show that $f$ is differentiable at 0, and find $f'(0)$.

**6.** *(15 points)* Use the Mean Value Theorem to prove that

$$(1+x)^{1/3} < 1 + \frac{x}{3} \quad \text{for } x > 1.$$

[*Hint:* Use the fact that the derivative of $(1+x)^{1/3}$ is $\frac{1}{3}(1+x)^{-2/3}$.]

(1) (first method) Let $\varepsilon > 0$ be given. Since $\lim_{x\to c} g(x) = 0$, (2)

[15] then there exists $\delta > 0$ such that if $0 < |x - c| < \delta$, then $|g(x) - 0| < \frac{\varepsilon}{M}$. (5) Therefore, if $0 < |x-c| < \delta$, then

$$|f(x)g(x) - 0| \overset{(2)}{=} |f(x)g(x)| \overset{(2)}{=} |f(x)|\cdot|g(x)| \overset{(2)}{<} M \cdot \frac{\varepsilon}{M} = \varepsilon.$$

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(alternate method) (2) Let $(x_n)$ be given such that $x_n \neq c$ for all $n$, (2) and $\lim(x_n) = c$. Since $\lim_{x\to c} g(x) = 0$, then $\lim(g(x_n)) = 0$ by the sequential criterion for limits. (2) Since $|f(x)| \leq M$ for all $x \in \mathbb{R}$, then $|f(x_n)| \leq M$ for all $n$, so $(f(x_n))$ is a bounded sequence. Since $(f(x_n))$ is bounded and $\lim(g(x_n)) = 0$, it follows (5) from a homework problem done in class that $\lim(f(x_n)\cdot g(x_n)) = 0$. So, by the sequential criterion, $\lim_{x\to c}(f(x)\cdot g(x)) = 0$. (4)

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(2) (first method) Let $L = \lim_{x\to 0} f(x)$, we want to prove $\lim_{x\to 0} g(x) = L$.

[15] (2) Let $\varepsilon > 0$ be given. Then there exists $\delta > 0$ such that if $0 < |x - 0| < \delta$, then $|f(x) - L| < \varepsilon$. (2) Let $\tilde{\delta} = \frac{\delta}{2}$. (5)

If $0 < |x - 0| < \tilde{\delta}$, (2) then $0 < |x| < \frac{\delta}{2}$, (2) so $0 < |2x| < \delta$, so $0 < |2x - 0| < \delta$, so $|f(2x) - L| < \varepsilon$, so $|g(x) - L| < \varepsilon$. (2)

This proves that for all $\varepsilon > 0$ there exists $\tilde{\delta} > 0$ such that if $0 < |x - 0| < \tilde{\delta}$ then $|g(x) - L| < \varepsilon$. So $\lim_{x\to 0} g(x) = L$.

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(alternate method). Let $(x_n)$ (2) be given such that $x_n \neq 0$ for all $n$, and $\lim(x_n) = 0$. Let $(y_n) = (2x_n)$. (5) Then $y_n \neq 0$ for all $n$, and $\lim(y_n) = \lim(2x_n) = 2\lim(x_n) = 0$. (2) So by the sequential criterion for limits, $\lim(f(y_n)) = L$. (2) Therefore $\lim(f(2x_n)) = L$, so $\lim(g(x_n)) = L$. (2) So, by the sequential criterion, $\lim_{x\to 0} g(x) = L$. (2)

(3) (a) Let $g(x) = x^2$. Then $g$ is a polynomial, so $g$ is continuous on $\mathbb{R}$, (3)

[10] so $\lim_{x\to 3} g(x) = g(3) = 9$. ~~But~~ Also $g(x) = f(x)$ for all $x \neq 3$,

(2) (2)

so $\lim_{x\to 3} f(x) = \lim_{x\to 3} g(x)$. But $f(3) = 7$, so $\lim_{x\to 3} f(x) \neq f(3)$,

so $f$ is not continuous at 3. (2)

(Note: as usual, there are a number of other ways to prove this)

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(b) We saw in class that if a function is differentiable at a point then it must be continuous at that point.
(5)
Since $f$ is not continuous at 3, it cannot be differentiable at 3.

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4. (a) Let $\varepsilon > 0$ be given. Let $\delta = \varepsilon$. (3) Suppose $|x - 0| < \delta$.

[10] If $x \in \mathbb{Q}$ then $f(x) = x$, so $|f(x) - 0| = |x - 0| < \delta = \varepsilon$

If $x \notin \mathbb{Q}$ then $f(x) = -x$, so $|f(x) - 0| = |-x - 0| = |x| = |x - 0| < \delta = \varepsilon$. (4)

So in all cases, $|f(x) - 0| < \varepsilon$. Since $f(0) = 0$, this means

$|f(x) - f(0)| < \varepsilon$. So $f$ is continuous at $0$. (3)

~~Alternatively, we could~~

[15] (b) Let $g(x) = \dfrac{f(x) - f(0)}{x - 0}$ (3) for $x \neq 0$. Then

for $x \in \mathbb{Q}$, $g(x) = \dfrac{x - 0}{x - 0} = 1$ ; and for $x \notin \mathbb{Q}$, $g(x) = \dfrac{-x - 0}{x - 0} = \dfrac{-x}{x} = -1$.

So $g(x) = \begin{cases} 1 & \text{for } x \in \mathbb{Q} \\ -1 & \text{for } x \notin \mathbb{Q} \end{cases}$ (6) We proved in class that

this function does not have a limit at 0 (to see this, you can use the sequential criterion with $(x_n) = \dfrac{(-1)^n}{n}$ and $g(x_n) = (-1)^n$) (3) Since $\lim_{x\to 0} g(x)$ does not exist,

then $\lim_{x\to 0} \dfrac{f(x) - f(0)}{x - 0}$ does not exist, so $f$ is not differentiable at $0$. (3)

(5) Let $g(x) = \frac{f(x) - f(0)}{x - 0}$ for $x \neq 0$. Then
[15]

(3)

$g(x) = \frac{x^2 \sin(\frac{1}{x}) - 0}{x - 0} = x \sin(\frac{1}{x})$, for all $x \neq 0$. We claim $\lim_{x \to 0} g(x) = 0$

(3)

To see this, let $\varepsilon > 0$ be given. Let $\delta = \varepsilon$. If $0 < |x - 0| < \delta$,

Then $|g(x) - 0| = |x \sin(\frac{1}{x})| = |x| |\sin \frac{1}{x}| \leq |x| \cdot 1 = |x|$. (4)

Since $|g(x) - 0| \leq |x|$ and $|x| < \delta = \varepsilon$, Then $|g(x) - 0| < \varepsilon$.

This proves $\lim_{x \to 0} g(x) = 0$, (2) so $\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = 0$, so $f'(0) = 0$.

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(6.) (see next page) $\rightarrow$

6. The problem as originally stated was correct (though perhaps a little misleading). The problem as I amended it at the exam was incorrect. I decided to award students full credit for the problem who correctly stated the Mean Value Theorem and applied it to the function suggested by the problem. (Eleven of fifteen students got full credit on this problem; the other four got scores ranging from 10 to 13 out of 15.)

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What I meant to ask was the following:

Use the Mean Value Theorem to prove that

$$(1+x)^{1/3} < 1 + \frac{x}{3} \quad \text{for } x > 0.$$

Proof: Let $f(x) = (1+x)^{1/3}$ (you could also use $f(x) = (1+\frac{x}{3} - (1+x)^{1/3})$) and apply the Mean Value Theorem on $[0, x]$. This gives that there exists $c \in (0, x)$ such that $f'(c) = \frac{f(x)-f(0)}{x-0}$,

or $\frac{1}{3}(1+c)^{-2/3} = \frac{(1+x)^{1/3}-1}{x}$. Since $c > 0$, then $1+c > 1$,

so $(1+c)^{2/3} > 1$, so $(1+c)^{-2/3} = \frac{1}{(1+c)^{2/3}} < 1$, so $\frac{1}{3}(1+c)^{-2/3} < \frac{1}{3}$.

Therefore $\frac{(1+x)^{1/3}-1}{x} < 1/3$, so $(1+x)^{1/3} - 1 < \frac{x}{3}$, so $(1+x)^{1/3} < 1 + \frac{x}{3}$.

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During the exam I amended the problem to read: use the Mean Value Theorem to prove that $(1+x)^{1/3} < 1 + \frac{x-1}{3}$ for $x > 1$. This inequality is false, as can be seen by taking $x$ just slightly larger than 1 (e.g. $x = 1.1$), for then $(1+x)^{1/3} \approx 2^{1/3}$ and $1 + \frac{x-1}{3} \approx 1$, and $2^{1/3} \not< 1$.

If you apply the Mean Value Theorem to $f(x) = (1+x)^{1/3}$ on the interval $[1, x]$, you can prove that $(1+x)^{1/3} < 2^{1/3} + \frac{x-1}{3}$ for $x > 1$. Several people did this on their exams.