Theorem. If $\lim a_n = A$ and $\lim b_n = B$, then $\lim a_n b_n = AB$.

Proof. Let $\epsilon > 0$ be given.

Since the sequence (b_n) converges, then it is bounded. Therefore there exists a real number M such that, for every natural number n, we have $|b_n| \leq M$. Define M_1 to be the larger of M and 1. Since $M_1 \geq 1$ then $M_1 > 0$, and for every natural number n we have $|b_n| \leq M_1$, so $|b_n|/M_1 \leq 1$.

Since $\lim a_n = A$, then by definition of limit, there exists $K_1 \in \mathbf{N}$ such that for all $n \geq K_1$, $|a_n - A| < \epsilon/2M_1$.

Define M_2 to be the larger of $|A|$ and 1. Since $M_2 \geq 1$ then $M_2 > 0$, and we have $|A|/M_2 \leq 1$.

Since $\lim b_n = B$, then by definition of limit, there exists $K_2 \in \mathbf{N}$ such that for all $n \geq K_1$, $|b_n - B| < \epsilon/2M_2$.

Now let K be the larger of K_1 and K_2. For every natural number n greater than K, we have

$$
\begin{aligned}
|a_n b_n - AB| &= |a_n b_n - Ab_n + Ab_n - AB| \\
&\leq |a_n b_n - Ab_n| + |Ab_n - AB| \\
&= |a_n - A||b_n| + |b_n - B||A| \\
&< \left(\frac{\epsilon}{2M_1}\right)|b_n| + \left(\frac{\epsilon}{2M_2}\right)|A| \\
&= \frac{\epsilon}{2}\left(\frac{|b_n|}{M_1} + \frac{|A|}{M_2}\right) \\
&\leq \frac{\epsilon}{2}(1+1) = \epsilon,
\end{aligned}
$$

which proves that

$$|a_n b_n - AB| < \epsilon.$$