Math 5403 — Calculus of Variations
Assignment 1

1. For $f \in C[0,1]$, define

$$\|f\| = \left(\int_0^1 |f(x)|^2 \, dx \right)^{1/2}.$$

Show that this defines a norm on the linear space $C[0,1]$.

2. Show that if we define J on $C^1[0,1]$ by

$$J[y] = \int_0^1 \sqrt{1+(y')^2} \, dx,$$

then at $y(x) = ax + b$, the first differential of J at y is given by

$$\delta J[h] = \frac{a}{\sqrt{1+a^2}} \left(h(1) - h(0) \right).$$

3. Suppose that J is a functional on a linear space X and that at some $y_0 \in X$ and $h \in X$, the first differential $\delta J[h]$ exists. Show that then the first differential of the functional K defined by $K[y] = e^{J[y]}$ also exists at y_0 in the direction h, and find a formula for it.

4. Suppose $K(x_1, x_2)$ is a fixed function of two variables defined for $(x_1, x_2) \in [0,1] \times [0,1]$. For $y \in C[0,1]$, define

$$J[y] = \int_0^1 \int_0^1 K(x_1, x_2) y(x_1) y(x_2) \, dx_1 \, dx_2.$$

Find the first differential $\delta J[h]$, where $h \in C[0,1]$.