Math 5403 — Calculus of Variations
Assignment 2

1. Define a functional J on $C^1[0,1]$ by

$$J[y] = \int_0^1 (y')^2(1-y')\ dx,$$

and define $\hat{y} \in C^1[0,1]$ by setting $\hat{y}(x) = 0$ for all $x \in [0,1]$. We will be looking at values of $J[y]$ when y satisfies the boundary conditions $y(0) = 0$ and $y(1) = 0$.

a) Show that $\hat{y}$ is a weak local minimum for J, by showing that $J[y] \geq J[\hat{y}]$ for all y such that $\|y - \hat{y}\|_s < 1/2$.

b) Show that $\hat{y}$ is not a strong local minimum for J, even if we restrict the admissible variations to the set $S = \{h \in C^1[0,1] : h(0) = h(1) = 0\}$.

Hint: define h to be the piecewise linear function whose graph consists of the line segment connecting the point $(0,0)$ to the point $(1-\epsilon^2, -\epsilon/2)$, and the line segment connecting the point $(1-\epsilon^2, -\epsilon/2)$ to the point $(1,0)$. Show that if ϵ is small enough then $\|h\|_w \leq \epsilon$, and $J[\hat{y}+h] < J[\hat{y}]$. (Actually this is technically not enough to finish the problem, because this h is not in C^1 and therefore not admissible. But you can ignore this technical point if you like.)

c) Show that the only solution of the Euler-Lagrange equation for J which satisfies the given boundary conditions is $\hat{y} \equiv 0$. Conclude that J does not have any strong minimum for the given boundary conditions.

2. Find the general solution of the Euler-Lagrange equation for the functional

$$J[y] = \int_a^b x\sqrt{1+(y')^2}\ dx.$$

The answer should contain two arbitrary constants.

3. Find the general solution of the Euler-Lagrange equation for the following functionals:

a) $J[y] = \displaystyle\int_a^b \frac{\sqrt{1+(y')^2}}{y}\ dx,$

b) $J[y] = \displaystyle\int_a^b y'(1+x^2y')\ dx.$

4. A line in the xz-plane with slope m and passing through the origin is revolved around the z-axis to create a cone.

The cone can be given by the parametric equations

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = mr$$

where the parameters r and θ satisfy $r \geq 0$ and $0 \leq \theta \leq 2\pi$. A curve γ on the cone can be given by an equation $r = r(\theta)$ defining r as a function of θ. This problem asks you to the curve of shortest length on the cone connecting two given points (θ_a, r_a) and (θ_b, r_b). (Such a curve is called a *geodesic* on the cone.)

a) Express the length of γ as a functional of $r(\theta)$, of the form $J[r(\theta)] = \int_{\theta_a}^{\theta_b} f(\theta, r, \frac{dr}{d\theta})\ d\theta$.

b) Show that the solutions of the Euler-Lagrange equation for minimizers of J must satisfy the equation

$$(1+m^2)\alpha^2\left(\frac{dr}{d\theta}\right)^2 = r^2(r^2-\alpha^2),$$

where α is a constant.

c) Solve this equation to show that the general solution of the Euler-Lagrange equations is given by

$$r(\theta) = \alpha \sec\left(\frac{\theta+\beta}{\sqrt{1+m^2}}\right),$$

where α and β are arbitrary constants. (Hint: integrate the equation in part b) by separating variables and making the substitution $r = \alpha \sec u$.)

(Food for thought — not part of the assignment: can you give a geometric interpretation of the result in part c)? If you had to make a cone out of paper with geodesics drawn on it, how would you draw them?)