Solutions to Assignment 2

1.

a). If $\|y - \hat{y}\|_s < 1/2$, then $\|y\|_s < 1/2$, so $|y'(x)| < 1/2$ for all $x \in [0,1]$, and hence $1 - y'(x) > 1/2$ for all $x \in [0,1]$. Therefore $J[y] \geq \int_0^1 (y')^2 \cdot \frac{1}{2}\, dx \geq 0 = J[\hat{y}]$.

b). If we define h as in the hint, then $|h|$ takes its maximum value at the point $(1-\epsilon^2, -\epsilon/2)$, where $|h| = \epsilon/2$, so $\|h\|_w = \epsilon/2 < \epsilon$.

Also h' is equal to the constant $m_1 = \dfrac{-\epsilon/2}{1-\epsilon^2}$ on the interval $(0, 1-\epsilon^2)$, and h' is equal to the constant $m_2 = \dfrac{\epsilon/2}{\epsilon^2} = \dfrac{1}{2\epsilon}$ on the interval $(1-\epsilon^2, 1)$. So (ignoring the fact that h' does not exist at $x = 1-\epsilon^2$),

$$\begin{aligned} J[\hat{y} + h] = J[h] &= \int_0^{1-\epsilon^2} m_1^2(1-m_1)\, dx + \int_{1-\epsilon^2}^1 m_2^2(1-m_2)\, dx \\ &= (1-\epsilon^2)m_1^2(1-m_1) + \epsilon^2 m_2^2(1-m_2) \\ &= \frac{\epsilon^2}{4(1-\epsilon^2)}\left(1 + \frac{\epsilon}{2(1-\epsilon^2)}\right) + \frac{1}{4}\left(1 - \frac{1}{2\epsilon}\right). \end{aligned}$$

In the last expression, the limit of the first term is 0 as $\epsilon \to 0$, and the limit of the second term is $-\infty$. So $\lim_{\epsilon \to 0} J[\hat{y} + h] = -\infty$. In particular we have that $J[\hat{y} + h] < 0$ for sufficiently small values of ϵ.

We have shown that in every ϵ-neighborhood of 0 in the weak norm, there exists some h such that $J[\hat{y} + h] < J[\hat{y}]$. So $\hat{y}$ is not a strong local minimum.

c). Here $J[y] = \int_a^b F(x, y, y')\, dx$, where $F(x, y, y') = (y')^2(1-y')$. Every weak local minimum $y(x)$ must satisfy the Euler-Lagrange equation on $[a,b]$, and since F is independent of x the Euler-Lagrange equation implies $\frac{d}{dx}(F_{y'}) = 0$, which means that $F_{y'}$ is constant on $[a,b]$. Therefore $2(y') - 3(y')^2$ is constant on $[a,b]$. It follows that y' is constant on $[a,b]$, say $y' = C$ on $[a,b]$, and therefore $y = Cx + D$ on $[a,b]$. Since $y(0) = 0$ and $y(1) = 0$, then $D = 0$ and $C + D = 0$; so $C = D = 0$, and therefore $y(x) \equiv 0$ on $[a,b]$.

This proves that the only weak local minimum satisfying the boundary conditions is $y(x) \equiv 0$. Since every strong local minimum is a weak local minimum, and $y(x) \equiv 0$ is not a strong local minimum, there are no strong local minimums for these boundary conditions. (In fact it's not hard to show there are no strong local minimums for any other boundary conditions, either.)

2. Since $F(x, y, y') = x\sqrt{1 + (y')^2}$ is independent of y, solutions of the Euler-Lagrange equation satisfy

$$F_{y'} = \frac{xy'}{\sqrt{1 + (y')^2}} = C$$

for some constant C. Solving for y' we obtain

$$y' = \frac{C}{\sqrt{x^2 - C^2}},$$

and integrating gives

$$y = C \ln\left|x + \sqrt{x^2 - C^2}\right| + D,$$

where D is another arbitrary constant.

3.

a). Since $F(x, y, y') = \frac{\sqrt{1+(y')^2}}{y}$ is independent of x, solutions of the Euler-Lagrange equation satisfy

$$F - y' F_{y'} = C$$

for some constant C. The latter equation simplifies to

$$\frac{1}{y\sqrt{1+(y')^2}} = C,$$

and solving for y' we obtain

$$y' = \frac{\sqrt{1-C^2y^2}}{Cy}.$$

Separating variables and integrating with respect to x gives

$$C\int \frac{y\ dy}{\sqrt{1-C^2y}} = \int\ dx,$$

or

$$\frac{-\sqrt{1-c^2y^2}}{C} = x + D,$$

where D is a constant. Solving for y gives

$$y = \frac{\sqrt{1-(Cx+E)^2}}{C},$$

where C and $E = CD$ are arbitrary constants.

b). Since $F(x, y, y') = y'(1 + x^2y')$ is independent of y, solutions of the Euler-Lagrange equation satisfy

$$F_{y'} = 1 + 2x^2y' = C$$

where C is an arbitrary constant. Solving for y' we obtain

$$y' = \frac{D}{x^2},$$

where $D = (C-1)/2$ is also an arbitrary constant. Integrating gives

$$y = -D/x + E,$$

where E is another arbitrary constant.

4.

a). The length of the curve on the cone $x = r\cos\theta$, $y = r\sin\theta$, $z = mr$ given by $r = r(\theta)$ for $\theta_a \le \theta \le \theta_b$ is

$$\begin{aligned} J[r(\theta)] &= \int_{\theta_a}^{\theta_b} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 + \left(\frac{dz}{d\theta}\right)^2}\ d\theta \\ &= \int_{\theta_a}^{\theta_b} \sqrt{\left(\frac{dr}{d\theta}\cos\theta - r\sin\theta\right)^2 + \left(\frac{dr}{d\theta}\sin\theta + r\cos\theta\right)^2 + \left(m\frac{dr}{d\theta}\right)^2}\ d\theta \\ &= \int_{\theta_a}^{\theta_b} \sqrt{r^2 + (1+m^2)\left(\frac{dr}{d\theta}\right)^2}\ d\theta. \end{aligned}$$

b). Since $F(\theta, r, r') = \sqrt{r^2 + (1+m^2)(r')^2}$ is independent of θ, solutions of the Euler-Lagrange equation satisfy

$$F - r'F_{r'} = \alpha$$

for some constant α. So

$$\sqrt{r^2 + (1+m^2)(r')^2} - \frac{(r')^2(1+m^2)}{\sqrt{r^2 + (1+m^2)(r')^2}} = \alpha,$$

which simplifies to

$$r^2 = \alpha\sqrt{r^2 + (1+m^2)(r')^2}.$$

Squaring both sides and solving for r', we get

$$(1+m^2)\alpha^2(r')^2 = r^2(r^2 - \alpha^2).$$

c). Separating variables and integrating, we get

$$\alpha\sqrt{1+m^2}\int \frac{dr}{r\sqrt{r^2-\alpha^2}} = \int \, d\theta.$$

Substituting $r = \alpha \sec u$ leads to

$$\alpha\sqrt{1+m^2}\int \, du = \int \, d\theta,$$

or

$$\alpha\sqrt{1+m^2}\; u = \theta + \beta.$$

Therefore

$$r = \alpha \sec\left(\frac{\theta+\beta}{\sqrt{1+m^2}}\right).$$