Complex Analysis II
Exam 1

1. Suppose Ω is an open set in $\mathbf{C}$, and define

$$\mathcal{F} = \{f : f \text{ is holomorphic on } \Omega \text{ and } \mathrm{Re}(f(z)) \leq 0 \text{ for all } z \in \Omega\}.$$

Show that $\mathcal{F}$ is a normal family. (Hint: consider e^{-f}.)

2. Give an explicit function $f(z)$ which maps the set

$$\Omega = \{z : \mathrm{Im}(z) > 0 \text{ and } \mathrm{Re}(z) > 0\}$$

conformally onto the unit disc $\{|z| < 1\}$. Verify that your function is conformal. (Hint: first consider the effect of the map $g(z) = z^2$.)

3. Suppose Ω is a holomorphically simply connected open set in $\mathbf{C}$. Consider the statement: "For every pair of points P and Q in Ω, there exists a conformal self-map f of Ω such that $f(P) = Q$."

a. Prove the statement in case $\Omega = \mathbf{C}$.
b. Prove the statement in case $\Omega \neq \mathbf{C}$.

4. We define a function u on $D(0,1)$ to be *radial* if, for all $z = re^{i\theta} \in D(0,1)$, $u(z) = f(r)$, where $f(r)$ is a function which depends only on r, not on θ. Show that if u is radial and harmonic on $D(0,1)$, then u is constant on $D(0,1)$.

5. Suppose Ω is a disk in $\mathbf{C}$ and u is harmonic on Ω.

a. Show that $\dfrac{\partial u}{\partial x} - i\dfrac{\partial u}{\partial y}$ is holomorphic on Ω.

b. Show that if γ is a simple closed curve in Ω, then

$$\int_\gamma \left(-\frac{\partial u}{\partial y}\, dx + \frac{\partial u}{\partial x}\, dy \right) = 0.$$