Complex Analysis II
Final Exam (take-home)

1. Let $\mathbf{H}$ be the upper half-plane, $\mathbf{H} = \{x + iy : y > 0\}$. Suppose f is holomorphic on $\mathbf{H}$ and $f(\mathbf{H}) \subseteq \mathbf{H}$. Show that if $f(i) = i$, then $|f'(i)| \leq 1$.

2.a. Suppose $r > 0$ and $z_0 \in \mathbf{C}$. Show there exists a constant M (depending only on r) such that for every positive harmonic function u on $\{z : |z - z_0| < r\}$, we have $u(z_1) \leq Mu(z_2)$ for any two points z_1 and z_2 in $\{z : |z - z_0| < r/2\}$. (Hint: use Corollary 7.6.2)

b. Suppose Ω is a connected open set in $\mathbf{C}$, and K is a compact subset of Ω. Show there exists a constant M (depending only on K and Ω) such that for every positive harmonic function u on Ω, we have $u(z_1) \leq Mu(z_2)$ for any two points z_1 and z_2 in K.

3. It follows from what we proved in class that the following statements are true for every *bounded* open set Ω:

(i) if u is continuous on the closure of Ω and harmonic on Ω, and $u = 0$ on $\partial\Omega$, then $u \equiv 0$ on Ω.

(ii) if u and v are continuous on the closure of Ω with u harmonic on Ω and v subharmonic on Ω, and $v \leq u$ on $\partial\Omega$, then $v \leq u$ on Ω.

Show, by giving examples, that both of these statements are false for $\Omega = \{z = x+iy : 0 < x < 1\}$. (Hint: consider $\sin \pi z$.)

4. Prove that

$$\cos \pi z = \prod_{n=1}^{\infty} \left(1 - \frac{4z^2}{(2n-1)^2}\right)$$

5. Find product expansions for $\sinh z$ and $\cosh z$.

6. Suppose f and g are holomorphic on an open set Ω. Show there exist holomorphic functions f_1, g_1, and h on Ω such that $f = hf_1$ and $g = hg_1$ on Ω, and f_1 and g_1 have no common zeroes.

7. The power series

$$\sum_{j=1}^{\infty} \frac{z^j}{j}$$

defines a holomorphic function $f(z)$ on $\{|z| < 1\}$. Find an open set Ω in $\mathbf{C}$ such that (i) f has a holomorphic extension to Ω and (ii) f does not have a holomorphic extension to any set larger than Ω. (Hint: find a closed-form expression for $f(z)$.)