Exam 1: solutions

1. There were three difficulties with this problem. First, as stated it is incorrect, as is shown by the example $f_n(z) \equiv -n$. Second, even when corrected, its proof, while not too difficult, involves too many steps to make it suitable for a test problem. Third, the hint should have referred to e^f, not e^{-f}.

The problem was actually taken from Ahlfors' text, which uses a different definition of "normal" than the one given in our text. In Ahlfors' text, a family $\mathcal{F}$ of functions on a connected open set Ω is said to be a *normal family in the classical sense* if every sequence in $\mathcal{F}$ has either a subsequence which converges normally (i.e., compactly on compact subsets of Ω) to a holomorphic complex-valued function, or a subsequence which converges normally to ∞.

Let's prove a corrected version of the problem: If Ω is a connected open subset of $\mathbf{C}$, the family $\mathcal{F}$ of all analytic functions f on Ω satisfying Re $f \leq 0$ on Ω is normal in the classical sense.

To prove this, let f_n be a given sequence in $\mathcal{F}$. Then $|e^{f_n(z)}| \leq 1$ for all $n \in \mathbf{N}$ and all $z \in \Omega$, so by Montel's Theorem, by passing to a subsequence we can assume that e^{f_n} converges normally to a holomorphic function g on Ω. Also, since e^{f_n} is never zero on Ω, it follows from Hurwitz' Theorem that either g is never zero on Ω or $g \equiv 0$ on Ω.

In case $g \equiv 0$ on Ω, then $|e^{f_n}|$ converges normally to the constant function 0, so $|\text{Re } f_n| = |\log |e^{f_n}||$, and therefore also f_n, converge normally to ∞. Hence from now on we can assume that g is never 0 on Ω.

Let Ω_1 be the set of all $P \in \Omega$ such that f_n converges uniformly on some neighborhood of P to some complex-valued holomorphic function; and let Ω_2 be the set of all $P \in \Omega$ such that f_n converges uniformly on some neighborhood of P to ∞. We claim that $\Omega_1 \cup \Omega_2 = \Omega$.

Let $P \in \Omega$ be given. Since $g(P) \neq 0$, there exists a disk D_1 centered at $g(P)$ whose closure does not contain 0, and therefore we can define a holomorphic branch of the logarithm, which we'll denote by $\log z$, on $\overline{D_1}$.

Since g is continuous on Ω and e^{f_n} converges normally to g, there exists a disk D centered at P such that for all $z \in \overline{D}$ and all sufficiently large n, both $g(z)$ and $e^{f_n(z)}$ are contained in D_1. Therefore $\log(e^{f_n})$ is defined on $\overline{D}$ for large n, and since $\log z$ is uniformly continuous on $\overline{D_1}$ we have that $\log(e^{f_n})$ converges uniformly to $\log g$ on $\overline{D}$.

Now for all $z \in \overline{D}$ and all n sufficiently large,

$$f_n(z) = \log(e^{f_n(z)}) + 2\pi i \alpha_n(z)$$

for some integer $\alpha_n(z)$. Since f_n and $\log(e^{f_n})$ are continuous functions of z, so is $\alpha_n(z)$, and since $\alpha_n(z)$ is integer-valued, it must therefore be constant.

There are now two possibilities: either the sequence of integers $|\alpha_n|$ converges to ∞, or does not. In the first case, since $\log(e^{f_n(z)})$ converges uniformly to a bounded function g on $\overline{D}$, then f_n converges uniformly to ∞ on $\overline{D}$. Hence $P \in \Omega_2$. In the second case, by the Bolzano-Weierstrass theorem we can find a subsequence α_{n_k} which converges to some integer m, and therefore we must have $\alpha_{n_k} = m$ for all sufficiently large k. Then $f_{n_k} = \log(e^{f_{n_k}}) + 2\pi i m$ for large k, so f_{n_k} converges uniformly on $\overline{D}$ to $g + 2\pi i m$. Hence $P \in \Omega_1$. This shows $\Omega = \Omega_1 \cup \Omega_2$.

Since Ω_1 and Ω_2 are clearly open, and Ω is connected, it follows that either $\Omega_1 = \Omega$ or $\Omega_2 = \Omega$. By a standard compactness argument, this completes the proof.

2. The map $g(z) = z^2$ takes $re^{i\theta}$ to $r^2 e^{2i\theta}$ and therefore is a one-to-one map from $\Omega = \{\text{Re } z > 0,\ \text{Im } z > 0\} = \{r > 0,\ 0 < \theta < \pi/2\}$ onto the upper half-plane $H = \{r > 0,\ 0 < \theta < \pi\}$. Also the Cayley transform $h(z) = (z - i)/(z + i)$ maps H conformally onto the open unit disk. So we can take $f(z) = h \circ g(z) = (z^2 - i)/(z^2 + i)$. Since g and h are conformal maps, so is f.

3a. The conformal self-maps of $\mathbf{C}$ are the linear maps $f(z) = az + b$ with $a \neq 0$, so we need to find $a \neq 0$ and b such that $aP + b = Q$. We can take $a = 1$ and $b = Q - P$.

3b. If Ω is the unit disk D, then we can take $f = \phi_{-Q} \circ \phi_P$, where $\phi_a(z) = (1-a)/(1-\bar{a}z)$. Since $\phi_a(a) = 0$ and $\phi_{-a}(0) = a$, it follows that $f(P) = Q$. If Ω is any holomorphically simply connected open set, then by the Riemann mapping theorem there exists a conformal map g from Ω onto D. By the above, there exists a conformal self-map h of D such that $h(g(P)) = g(Q)$, so we can define $f = g^{-1} \circ h \circ g$.

4. By the mean-value property of harmonic functions, we have that, for all $r \in (0,1)$, $u(0)$ equals the average value of u on the circle $C_r = \{|z| = r\}$. But since $u(z) = f(r)$ for all $z \in C_r$, then this average value is $f(r)$. In other words,

$$u(0) = \frac{1}{2\pi r} \int_{C_r} u \, ds = \frac{f(r)}{2\pi r} \int_{C_r} 1 \, ds = \frac{f(r)}{2\pi r} \cdot 2\pi r = f(r).$$

Therefore $u(re^{i\theta}) = f(r) \equiv u(0)$, so u is constant on D.

5a. Using subscripts to denote partial derivatives, we have that $(u_x)_x = (-u_y)_y$ on Ω because $u_{xx} + u_{yy} = 0$, and $(u_x)_y = -(-u_y)_x$ on Ω because, being C^2, u has equal mixed partial derivatives by Clairaut's theorem from calculus. So u_x and $-u_y$ satisfy the Cauchy-Riemann equations on Ω, and since u_x and $-u_y$ are C^1, this is enough to conclude that $u_x - iu_y$ is holomorphic on Ω.

5b. By part **a** and the Cauchy integral theorem, we know that

$$0 = \int_\gamma (u_x - iu_y) \, dz = \int_\gamma (u_x - iu_y)(dx + idy) = \int_\gamma (u_x \, dx + u_y \, dy) + i \int_\gamma (-u_y \, dx + u_x \, dy) .$$

Since a complex number is zero only if its real and imaginary parts are zero, then $\int_\gamma (-u_y \, dx + u_x \, dy) = 0$.