Math 5453
Test 1

1. *(20 points)* Prove that if f' is continuous on $[a,b]$ then f is of bounded variation on $[a,b]$ and $V[a,b] = \int_a^b |f'|\, dx$.

2. *(20 points)* Prove that if f is continuous on $[a,b]$ and ϕ is of bounded variation on $[a,b]$ then

$$\left| \int_a^b f \, d\phi \right| \le \left(\sup_{[a,b]} |f|\right) \cdot V[\phi; a, b].$$

(You may assume the existence of the integral $\int_a^b f \, d\phi$.)

3. *(20 points)* Suppose $f(x)$ is of bounded variation on $[a,b]$, and that $f(x) \in [p,q]$ for all $x \in [a,b]$. Prove that if g is a Lipschitz function on $[p,q]$, then $g(f(x))$ is of bounded variation on $[a,b]$.

4. *(10 points)* Suppose f is continuous on $[a,b]$, $f(x) \ge 0$ for all $x \in [a,b]$, and ϕ is increasing on $[a,b]$. Prove that $\int_a^b f \, d\phi \ge 0$.

5. *(10 points)* Suppose $f(x) = x^2$ on $[0,1]$ and $\phi(x) = 3$ on $[0,1]$. Find $\int_0^1 f \, d\phi$ (and prove your answer).

6. *(20 points)* Suppose f is continuous on $[a,b]$, ϕ is continuous on $[a,b]$, and ϕ is of bounded variation on $[a,b]$. Prove that

$$\int_a^b f \, d(\phi^2) = \int_a^b f(2\phi) \, d\phi.$$