Math 5453
Test 2

1. *(20 points)* Prove that the union $E = \cup E_k$ of a countable number of measurable sets is measurable, with $|E| \leq \sum |E_k|$.

2. *(15 points)* Prove that if f is measurable and finite a.e. on a set E, and ϕ is continuous on $\mathbf{R}$, then $\phi(f)$ is measurable on E.

3. *(15 points)* Prove that if f is measurable on $\mathbf{R}^n$ and $\mathbf{h}$ is a fixed element of $\mathbf{R}^n$, then the function $g(\mathbf{x})$ defined by

$$g(\mathbf{x}) = f(\mathbf{x} + \mathbf{h})$$

is measurable on $\mathbf{R}^n$.

4. *(15 points)* Suppose $|E| < \infty$, and $\{f_k\}$ is a decreasing sequence of measurable functions on E such that $\lim_{k\to\infty} f_k(x) = 0$ for all $x \in E$. For each k define

$$E_k = \{\mathbf{x} \in E : f_k(\mathbf{x}) > 1\}.$$

Prove that $\lim_{k\to\infty} |E_k| = 0$.

5. *(15 points)* Suppose f and g are continuous functions defined on an open subset G of $\mathbf{R}^n$, and suppose $f = g$ almost everywhere on G. Prove that $f = g$ everywhere on G.

6. *(20 points)* Suppose a is a given real number such that $0 < a < 1$.

a. Suppose $E \subset \mathbf{R}$ is measurable and $|E| > 0$. Prove there exist disjoint open intervals $\{I_k\}$ such that $E \subset \cup_{k=1}^{\infty} I_k$ and

$$\sum_{k=1}^{\infty} |I_k| < \left(\frac{1}{a}\right) |E|.$$

b. Prove that if $E \subset \mathbf{R}$ is measurable and $|E| > 0$, then there exists some open interval I in $\mathbf{R}$ such that $|E \cap I| > a|I|$. (Hint: assume the contrary and use part a.)