Math 5453
Final Exam

1. *(15 points)*

a. State and prove Fatou's Lemma for non-negative measurable functions. You may assume the Monotone Convergence Theorem.

b. State and prove the Dominated Convergence Theorem for non-negative measurable functions. (Use Fatou's Lemma.)

2. *(15 points)* Suppose f is a measurable function on E and $a < f(\mathbf{x}) \leq b$ for all $\mathbf{x} \in E$, where a and b are finite real numbers. Let $\omega(\alpha)$ be the distribution function for f on E. Prove that

$$\int_E f = -\int_a^b \alpha \, d\omega(\alpha).$$

3. *(15 points)*

a. Let $f(x) = x^{-1/2}$ on $E = [0, 1]$. Compute the distribution function $\omega(\alpha)$ of f. (Find simple formulas for $\omega(\alpha)$ which define it piecewise on the domain $\alpha \in (-\infty, \infty)$.)

b. We proved in class that if f is finite a.e. and measurable on E, and either $\int_E f$ or $\int_{-\infty}^{\infty} \alpha \, d\omega(\alpha)$ is finite, then the other exists and is finite, and

$$\int_E f = -\int_{-\infty}^{\infty} \alpha \, d\omega(\alpha).$$

Using this fact and your answer to part **a.**, find the integral of $f(x) = x^{-1/2}$ on $E = [0, 1]$. Justify each step of your computation.

4. *(15 points)*

a. Suppose f and g are measurable functions on E and $0 \leq f \leq g$ on E. Prove that $\int_E f \leq \int_E g$.

b. Suppose f is a measurable function on $\mathbf{R}$ and f is continuous at 0. Prove that

$$\lim_{n \to \infty} n \int_{[0, \frac{1}{n}]} f = f(0).$$

(Hint: use part **a.**)

5. *(15 points)* Suppose f is continuous on $\mathbf{R}$ and $f \in L(\mathbf{R})$. Prove that

$$\lim_{n \to \infty} \int_{[0,1]} \left| f\left(x + \frac{1}{n}\right) - f(x) \right| \, dx = 0.$$

6. *(10 points)* Suppose E_k are measurable subsets of E such that $\lim_{k \to \infty} |E_k| = 0$ and $E_{k+1} \subset E_k$ for all k. Prove that if $f \in L(E)$ then $\lim_{k \to \infty} \int_{E_k} f = 0$.

7. *(15 points)* Suppose $\{f_n\}$ is a sequence of measurable functions defined on a measurable subset E of $\mathbf{R}^n$. Let

$$S = \{\mathbf{x} \in E : \text{ the sequence } \{f_n(\mathbf{x})\} \text{ converges}\}.$$

(Here "converges" means "converges to a finite limit".) Prove that S is measurable.