Comment on second problem on Assignment 5

The second problem on Assignment 5 considered the situation in which a function f was measurable with respect to a σ-algebra Σ on a set X, but not necessarily measurable with respect to another, smaller, σ-algebra Σ_0 contained in Σ. Given that f is integrable on (X, Σ, μ), the problem defined the *conditional expectation* of f with respect to Σ_0 to be another function f_0 which has the property that, for all Σ_0-measurable functions g, we have

$$\int_X fg \, d\mu = \int_X f_0 g \, d\mu,$$

whenever both integrals exist and are finite.

A question that came up in some people's solution to this problem was whether f_0 must necessarily equal f almost everywhere. The answer to this question is no. Here is a typical example to illustrate this. Suppose X is the interval $[0, 2]$ in the real line, and Σ is the σ-algebra of Lebesgue measurable sets in X. Let Σ_0 be the σ-algebra consisting of all sets of the form $A \cup B$, where A is a Lebesgue measurable set in $[0, 1]$, and either $B = \emptyset$ or $B = [1, 2]$. You should check that Σ_0 is indeed a σ-algebra. Then answer the following questions:

(i) Is the function $f(x) = x$ measurable with respect to Σ_0?
(ii) In general, which functions are measurable with respect to Σ_0?
(iii) If $f(x) = x$ on $[0, 2]$, then what function f_0 is the conditional expectation of f with respect to Σ_0?
(iv) More generally, given an integrable Lebesgue measurable function f on $[0, 2]$, what will be its conditional expectation f_0 with respect to Σ_0?

For an explanation of what the conditional expectation, as defined above, has to do with the conditional expectation you learned about in your introductory probability class, you can check out the article "Conditional Expectation" by Jason Swanson at

http://math.swansonsite.com/instructional/condexp.pdf