Math 5463 — Spring 2013
Exam 1

(Note: For these problems, you may cite any result that we have done in class, without having to prove it.)

1. Suppose $\{f_k\}$ is a sequence of functions in $L^p(E)$, where $1 \le p < \infty$, and suppose g is a function in L^q, where q is the conjugate to p. Show that if $\{f_k\}$ converges to f in $L^p(E)$ norm, then $\lim_{k\to\infty} \int_E f_k g = \int_E fg$.

2. Let X be the set of all differentiable functions on $[0,1]$, and for f in X, define

$$\|f\| = \left(\int_0^1 f'(x)^2\, dx\right)^{1/2}.$$

For each of the following statements, either give a (short) proof that the statement is true, or give an example showing that the statement is false.

(i) For all f and g in X, $\|f+g\| \le \|f\| + \|g\|$.
(ii) For all $c \in \mathbf{R}$ and f in X, $\|cf\| = |c| \cdot \|f\|$.
(iii) For all f in X, $\|f\| \ge 0$.
(iv) If f is in X and $\|f\| = 0$, then f is the zero function in X.

3. Consider the following statement: If f is in $L^p(E)$ and $\int_E fg = 0$ for all g in L^q, where q is the conjugate to p, then $f = 0$ almost everywhere on E.

a. Prove the statement in the case $1 < p < \infty$.

b. Prove the statement in the case $p = 1$.

4. Suppose T is a map from $L^p(\mathbf{R})$ to $\mathbf{R}$ such that (i) $T(f+g) = T(f) + T(g)$ for all f and g in $L^p(\mathbf{R})$ and (ii) $|T(f)| \le \|f\|_p$ for all f in $L^p(\mathbf{R})$. Show that if $T(f) = 0$ for every continuous function f with compact support, then $T(f) = 0$ for all f in $L^p(\mathbf{R})$. (Here $1 \le p < \infty$).

5. Suppose $\{f_k\}$ is a sequence of functions in $L^p(E)$, where $1 \le p < \infty$. Suppose $\{f_k\}$ converges pointwise on E to a function g, and $\{f_k\}$ converges in $L^p(E)$ norm to a function h. Show that $g = h$ almost everywhere on E.

(1.) For all $k \in \mathbb{N}$, $\left|\int_E f_k g - \int_E fg\right| = \left|\int_E (f_k - f)g\right| \leq$

$\leq \int_E |f_k - f| \cdot |g| \leq \|f_k - f\|_p \cdot \|g\|_q$. Since $\|f_k - f\| \to 0$, it follows that $\left|\int_E f_k g - \int_E fg\right| \to 0$. So $\lim_{k\to\infty} \int_E f_k g = \int_E fg$.

(note: the restriction $p < \infty$ was unnecessary: the proof also works for $p = \infty$.)

(2.) (i) True: $\|f+g\| = \|f' + g'\|_{L^2} \leq \|f'\|_{L^2} + \|g'\|_{L^2}$ (by Minkowski's Inequality)

$= \|f\| + \|g\|$.

(ii) True: $\|cf\| = \left[\int_0^1 (cf')^2 dx\right]^{\frac{1}{2}} = |c|\left[\int_0^1 (f')^2\right]^{\frac{1}{2}} = |c| \cdot \|f\|$.

(iii) True: $(f')^2 \geq 0$ on $[0,1]$, so $\int_0^1 (f')^2 \geq 0$, so $\|f\| \geq 0$

(iv) False: Let $f \equiv 1$ on $[0,1]$, for example. Then $f' \equiv 0$, so $\|f\| = \left[\int_0^1 (f')^2\right]^{1/2} = 0$, but f is not the zero function.

(3.) Let $g = (\operatorname{sign} f)\,|f|^{p-1}$. Then $g \in L^q$ (as seen in class)

so $0 = \int_E fg = \int_E |f|^p$, which implies $|f|^p \equiv 0$ a.e. (since $|f|^p \geq 0$ on E), and hence $f = 0$ a.e.

(This argument works as well for $p=1$ as for $p \in (1,\infty)$. However, proving that $g \in L^q$ is a little different ~~when~~ for $p = 1$ than for $p \in (1,\infty)$. In case $p=1$, then $q = \infty$, and $g \in L^\infty$ because $|g| \leq 1$ everywhere. In case $p \in (1,\infty)$, $q = \frac{p}{p-1}$, and $\int_E |g|^q = \int_E |f|^p$, so $\|g\|_q = \|f\|_p < \infty$.)

(4.) Let $f \in L^p(\mathbb{R})$ be given. Since continuous functions with compact support are dense in $L^p(\mathbb{R})$ for $1 \le p < \infty$, Then for every $\varepsilon > 0$ there exists $\tilde{f}$ which is continuous with compact support and satisfies $\| f - \tilde{f} \|_p < \varepsilon$.

So
$$|T(f)| = |T(\tilde{f} + f - \tilde{f})| = |T(\tilde{f}) + T(f - \tilde{f})|$$
$$= |0 + T(f - \tilde{f})|$$
$$\le \| f - \tilde{f} \|_p < \varepsilon .$$

Since ε was arbitrary, This proves $T(f) = 0$.

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(5) We know from class that since f_k converges in L^p to h, Then there exists some subsequence $\{f_{k_i}\}$ which converges pointwise a.e. to h.

(In fact, if f_k converges to h in L^p, Then f_k converges in measure to h, and so has a subsequence which converges pointwise a.e. to h). Since f_k converges pointwise to g, Then all of its subsequences, including $\{f_{k_i}\}$ must also converge pointwise to g. Since The limit of a sequence of real numbers is unique, it follows that $h(x) = g(x)$ for almost every $x \in E$.