Math 5463 — Spring 2013
Final Exam

1. Suppose (X, μ) is a measure space and f_n are functions in $L^p(X, \mu)$ which converge pointwise to a function f. Suppose also that there exists a constant $M < \infty$ such that $\|f_n\| \leq M$ for all $n \in \mathbf{N}$. Show that f is in $L^p(X, \mu)$.

2. Suppose (X, Σ, μ) is a measure space with $\mu(X) < \infty$. Show that if f is any function in $L^8(X, \mu)$, then f is also in $L^7(X, \mu)$ and

$$\|f\|_{L^7} \leq \|f\|_{L^8} \cdot \mu(X)^{1/56}.$$

(Hint: use Hölder's inequality.)

3. Suppose $\{e_n\}$ are orthonormal vectors in a Hilbert space H, and x is an element of H. Show that if $\epsilon > 0$, and S is the set of all numbers n for which $|\langle x, e_n \rangle| > \epsilon$, then S can contain no more than $\|x\|^2/\epsilon^2$ elements.

4. Suppose ϕ is an additive set function on a measure space (X, Σ), and V is its total variation. (In Royden and Fitzpatrick, V is denoted by $|\phi|$.) Define $\mu = \frac{1}{2}(V + \phi)$ and $\nu = \frac{1}{2}(V - \phi)$. Show that $\mu \perp \nu$.

5. Let $X = [0, 2]$ and define a σ-algebra Σ of subsets of X by $\Sigma = \{\emptyset, [0, 1], (1, 2], [0, 2]\}$.

(i) Define $f : [0, 2] \to \mathbf{R}$ by $f(x) = x$. Is f measurable with respect to Σ? Prove your answer.

(ii) Suppose λ is Lebesgue measure on X, and define a measure μ on (X, Σ) by $\mu(E) = \int_E f \, d\lambda$. Find, with proof, the Radon-Nikodym derivative of μ with respect to λ. That is, find the function f_0 which is measurable on (X, Σ) and satisfies $\mu(E) = \int_E f_0 \, d\lambda$ for all E in Σ.

6. In the measure space $(\mathbf{R}, \mathcal{M})$, where $\mathcal{M}$ is the Lebesgue σ-algebra, let λ denote Lebesgue measure and define measures μ and ν by

$$\mu(E) = \lambda(E \cap [0, 2]) \quad \text{and} \quad \nu(E) = \lambda(E \cap [1, 3])$$

for all measurable sets E. The Lebesgue decomposition theorem says that there exists a measurable function f and a set Z with $\nu(Z) = 0$ such that for all measurable sets E,

$$\mu(E) = \int_E f \, d\nu + \mu(E \cap Z).$$

Find this function f and set Z.

7. Let X be a set and let μ^* be an outer measure on the subsets of X. Show that if $\mu^*(E) = 0$, then E is μ^*-measurable.

8. Let X be a set and let $\mathcal{C}$ be the collection of all subsets of X which contain just one element. For each set $\{x\}$ in $\mathcal{C}$, define $\mu(\{x\}) = 1$.

(i) Find the outer measure μ^* induced by μ.

(ii) Which sets are μ^*-measurable?