(1) By Fatou's Lemma, $\int_X |f|^p \, d\mu = \int_X \lim_{n\to\infty} |f_n|^p \, d\mu \le \liminf_{n\to\infty} \int_X |f_n|^p \, d\mu$

$\le M < \infty$. So $\int_X |f|^p \, d\mu < \infty$, so $f \in L^p(X,\mu)$.

---

(2) Let $p = \frac{8}{7}$ and $q = 8$. Then $\frac{1}{p} + \frac{1}{q} = 1$, so we can apply Hölder's Inequality $\int_X |g h| \, d\mu \le \left(\int_X |g|^p d\mu\right)^{\frac{1}{p}} \left(\int_X |h|^q d\mu\right)^{\frac{1}{q}}$ with $g = |f|^7$ and $h = 1$. There results

$$\int_X |f|^7 \cdot 1 \, d\mu \le \left(\int_X \left(|f|^7\right)^{8/7} d\mu\right)^{7/8} \left(\int_X 1^8 \, d\mu\right)^{\frac{1}{8}} =$$

$$= \left(\int_X |f|^8 \, d\mu\right)^{7/8} \left(\mu(X)\right)^{\frac{1}{8}},$$ or $\|f\|_7^7 \le \|f\|_8^7 \, \mu(X)^{\frac{1}{8}}$. Taking the $\frac{1}{7}$ power of both sides gives $\|f\|_7 \le \|f\|_8 \, \mu(X)^{1/56}$.

---

(3) Let k be the number of elements in S.

Since $$\sum_{n=1}^{\infty} |\langle x, e_n\rangle|^2 \le \|x\|^2,$$

and $$\sum_{n=1}^{\infty} |\langle x, e_n\rangle|^2 \ge \sum_{n\in S} |\langle x, e_n\rangle|^2 \ge k\,\varepsilon^2,$$

Then $k\varepsilon^2 \le \|x\|^2$, so $k \le \|x\|^2/\varepsilon^2$.

---

(4) By the Jordan and Hahn decomposition Theorems, we can write $X = P \cup N$ where $P \cap N = \emptyset$, P and N are in Σ, and $\varphi^+(E) = \varphi(E \cap P)$ and $\varphi^-(E) = \varphi(E \cap N)$ for all E in Σ, where $\varphi = \varphi^+ - \varphi^-$ and $V = \varphi^+ + \varphi^-$.

Then $\mu = \frac{1}{2}(V + \varphi) = \frac{1}{2}(2\varphi^+) = \varphi^+$ and

$\nu = \frac{1}{2}(V - \varphi) = \frac{1}{2}(2\varphi^-) = \varphi^-$.

Since $\varphi^-(P) = \varphi(P \cap N) = \varphi(\emptyset) = 0$,
and for all $E \subseteq P^c$, $\varphi^+(E) = \varphi(E \cap P) = \varphi(\emptyset) = 0$,
Then $\varphi^- \perp \varphi^+$, so $\mu \perp \nu$.

---

(5) (i) No, because if we take $\alpha = \frac{1}{2}$, Then
$\{x : f(x) > \alpha\} = \{x \in X : x > \frac{1}{2}\} = (\frac{1}{2}, 2]$, which is not in Σ

(ii) Let $f_0 = \begin{cases} \frac{1}{2} & \text{for } x \in [0,1] \\ \frac{3}{2} & \text{for } x \in (1,2] \end{cases}$. Then f_0 is measurable on X because for all $\alpha \in \mathbb{R}$, $\{x : f_0(x) > \alpha\}$ is either $\emptyset$ (if $\alpha \geq \frac{3}{2}$), or $(1,2]$ (if $\frac{1}{2} \leq \alpha < \frac{3}{2}$), or X (if $\alpha < \frac{1}{2}$)

We can check that $\mu(E) = \int_E f_0 \, d\lambda$ for all E in Σ by considering each of the four possibilities for E:

- if $E = \emptyset$, $\mu(E) = 0$ and $\int_E f_0 \, d\lambda = 0$.
- if $E = [0,1]$, $\mu(E) = \int_{[0,1]} f \, d\lambda = \int_0^1 x \, dx = \frac{1}{2} = \int_{[0,1]} \frac{1}{2} \, dx = \int_{[0,1]} f_0 \, d\lambda$
- if $E = (1,2]$, $\mu(E) = \int_{(1,2]} f \, d\lambda = \int_1^2 x \, dx = \frac{3}{2} = \int_{(1,2]} \frac{3}{2} \, dx = \int_{(1,2]} f_0 \, d\lambda$
- if $E = X$, $\mu(E) = \mu([0,1]) + \mu((1,2])$
$= \int_{[0,1]} f_0 \, d\lambda + \int_{(1,2)} f_0 \, d\lambda = \int_X f_0 \, d\lambda$.

(6) We ~~can~~ take $f = \chi_{[1,2]}$ and $Z = [0,1]$.

Then $\nu(Z) = \lambda([0,1] \cap [1,3]) = \lambda(\{1\}) = 0$, and for all measurable E,

$$\int_E f \, d\nu + \mu(E \cap Z) = \int_E \chi_{[1,2]} \, d\nu + \mu(E \cap [0,1]) =$$

$$= \nu(E \cap [1,2]) + \mu(E \cap [0,1]) = \lambda(E \cap [1,3] \cap [1,2]) + \lambda(E \cap [0,1] \cap [0,2])$$

$$= \lambda(E \cap [1,2]) + \lambda(E \cap [0,1])$$

$$= \lambda((E \cap [1,2]) \cup (E \cap [0,1]))$$ (because $E \cap [1,2]$ and $E \cap [0,1]$ are disjoint)

$$= \lambda(E \cap [0,2])$$

$$= \mu(E).$$

---

(7) Let A be any subset of $\underline{X}$. We have to show that $\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \sim E)$.

Since $A \cap E \subseteq E$ and μ^* is monotone, then $\mu^*(A \cap E) \leq \mu^*(E) = 0$, so $\mu^*(A \cap E) = 0$. Thus we have to show that $\mu^*(A) = \mu^*(A \sim E)$.

Since $A \sim E \subseteq A$, then $\mu^*(A) \geq \mu^*(A \sim E)$.

On the other hand, by subadditivity we have

$\mu^*(A) = \mu^*((A \sim E) \cup (A \cap E)) \leq \mu^*(A \sim E) + \mu^*(A \cap E) = \mu^*(A \sim E)$,

so $\mu^*(A) \leq \mu^*(A \sim E)$.

8. (i) We claim $\mu^*(E)$ equals the number of elements in E for all $E \subseteq X$; that is μ^* is counting measure on the subsets of X.

To see this, first we prove $\mu^*(E) \leq (\#\text{ of elts in } E)$. In case E is uncountable, there are no countable unions of sets E_k in $\mathcal{C}$ such that $E \subseteq \bigcup_{k=1}^{\infty} E_k$, so $\mu^*(E) = \infty$ (by the convention that the infimum of the empty set is ∞). In case E is countable, $E = \bigcup_{x \in E} \{x\}$ is a countable union of sets in $\mathcal{C}$ which cover E, so by def of μ^*,
$\mu^*(E) \leq \sum_{x \in E} \mu(\{x\}) = \sum_{x \in E} 1 = (\#\text{ of elts in } E)$.

On the other hand, if $\{E_k\}_{k \in \mathbb{N}}$ is a countable ~~[illegible]~~ collection of sets in $\mathcal{C}$ which covers E, then every x in E must be in some E_k, so for every $x \in E$ there is at least one "1" in the sum $\sum_{k \in \mathbb{N}} \mu(E_k)$. Hence $\sum_{k \in \mathbb{N}} \mu(E_k) \geq (\#\text{ of elts in } E)$. Since $\mu^*(E)$ is the infimum of all such sums, $\mu^*(E) \geq (\#\text{ of elts in } E)$.

(ii) We know that counting measure is a measure on $P(X)$, the collection of all subsets of X. So for every subset $E \subseteq X$, if A is any other subset of X, then
$$\mu^*(A) = \mu^*(A \sim E) + \mu^*(A \cap E)$$
holds by additivity of counting measure. But this means that every subset E of X satisfies the definition of μ^*-measurability.