Math 5463 — Spring 2013
Exam 2

(Note: For these problems, you may cite any result that we have done in class, without having to prove it.)

1. Suppose $\{\phi_n\}_{n\in\mathbf{N}}$ is an orthonormal basis of a Hilbert space H.

a) Show that $\|\phi_m - \phi_m\| = \sqrt{2}$ for all natural numbers m and n.
b) Show that $\{\phi_n\}$ has no subsequence which converges in the norm of H.
c) Show that there does not exist any x in H such that $\langle x, \phi_n\rangle = 1$ for all $n \in \mathbf{N}$.
d) Show that there does exist some x in H such that $\langle x, \phi_n\rangle = \frac{1}{n}$ for all $n \in \mathbf{N}$.

2. Suppose ϕ is an additive set function on a measure space (X, Σ); and $A_1, A_2, \ldots$ are disjoint measurable subsets of X. Show that

$$\sum_{k=1}^{\infty} |\phi(A_k)| < \infty.$$

(Hint: Use the Jordan decomposition of ϕ.)

3. Let $\mathbf{N} = \{1, 2, 3, \ldots\}$ be the set of natural numbers, let Σ be the collection of all subsets of $\mathbf{N}$, and let μ be the counting measure on Σ. Define $f : \mathbf{N} \to \mathbf{R}$ by $f(k) = \dfrac{1}{2^k}$. Find (with proof) $\int_N f\, d\mu$.

4. Define functions $f(x)$ and $g(x)$ on $\mathbf{R}$ by

$$f(x) = \begin{cases} 0 \text{ for } x \le 0 \\ 1 \text{ for } x > 0 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 1 \text{ for } x \le 0 \\ 0 \text{ for } x > 0. \end{cases}$$

Let λ denote Lebesgue measure, and define measures μ_f and μ_g on the σ-algebra of Lebesgue measurable sets in $\mathbf{R}$ by setting $\mu_f(A) = \int_A f\, d\lambda$ and $\mu_g(A) = \int_A g\, d\lambda$ for measurable sets A. Say, with proof, whether each of the following statements is true or false.

a) $\mu_f << \lambda$
b) $\lambda << \mu_f$
c) $\mu_f \perp \mu_g$
d) $\mu_f << \mu_g$