(1.) a) $\|\varphi_m - \varphi_n\|^2 \overset{(2)}{=} \langle \varphi_m - \varphi_n, \varphi_m - \varphi_n \rangle \overset{(2)}{=} \langle \varphi_m, \varphi_m \rangle - \langle \varphi_n, \varphi_m \rangle - \langle \varphi_m, \varphi_n \rangle + \langle \varphi_n, \varphi_n \rangle$

[10]

$\overset{(2)}{=} \|\varphi_m\|^2 - 2\langle \varphi_n, \varphi_m \rangle + \|\varphi_n\|^2 \overset{(4)}{=} 1 + 0 + 1 = 2$, so $\|\varphi_m - \varphi_n\| = \sqrt{2}$.

[10] b) If $\{\varphi_n\}$ had a convergent subsequence, $\{\varphi_{n_k}\}$, then the subsequence would have to be Cauchy (3); that is, for every $\varepsilon > 0$ there would exist $K \in \mathbb{N}$ such that $\|\varphi_{n_k} - \varphi_{n_j}\| < \varepsilon$ for $k, j \geq K$. (4) But if $\varepsilon < \sqrt{2}$ this is impossible, by a). (3)

[10] c) If $x \in H$, then by Bessel's inequality, we have $\sum_{n=1}^{\infty} |\langle x, \varphi_n \rangle|^2 \leq \|x\|^2 < \infty$. (5) Therefore we cannot have $\langle x, \varphi_n \rangle = 1$ for all $n \in \mathbb{N}$, since $\sum_{n=1}^{\infty} 1 = \infty$. (5)

[10] d) Since $\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$, the sequence $(\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots)$ is in ℓ^2. (4) By the Riesz-Fischer Theorem, the map from H to ℓ^2 given by $x \mapsto (\langle x, \varphi_1 \rangle, \langle x, \varphi_2 \rangle, \langle x, \varphi_3 \rangle, \ldots)$ is an isomorphism (3); in particular it is onto. (4) So there exists $x \in H$ such that $(\langle x, \varphi_1 \rangle, \langle x, \varphi_2 \rangle, \langle x, \varphi_3 \rangle, \ldots) = (\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots)$.
(In fact, $x = \sum_{n=1}^{\infty} \langle x, \varphi_n \rangle \varphi_n$.)

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[18] (2.) The Jordan decomposition Theorem says that there exist finite measures $\overline{V}$ and $\underline{V}$ on (X, Σ) such that for all A in Σ, $\varphi(A) = \overline{V}(A) - \underline{V}(A)$. (3) Since $\overline{V}(A) \geq 0$ and $\underline{V}(A) \geq 0$ then $|\varphi(A)| = |\overline{V}(A) - \underline{V}(A)| \leq |\overline{V}(A)| + |\underline{V}(A)| = \overline{V}(A) + \underline{V}(A)$. (6)
Hence $\sum_{k=1}^{\infty} |\varphi(A_k)| \leq \sum_{k=1}^{\infty} \overline{V}(A_k) + \sum_{k=1}^{\infty} \underline{V}(A_k)$. Since $\overline{V}$ and $\underline{V}$
(3)

are measures, they are countably additive, so 3
$\sum_{k=1}^{\infty} \overline{V}(A_k) = \overline{V}(\bigcup_k A_k)$ and $\sum_{k=1}^{\infty} \underline{V}(A_k) = \underline{V}(\bigcup_k A_k)$. Since $\overline{V}$ and $\underline{V}$ are finite, both $\overline{V}(\bigcup_k A_k)$ and $\underline{V}(\bigcup_k A_k)$ are finite. 3

Hence $\sum_{k=1}^{\infty} |\nu(A_k)| < \infty$.

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3 One way to do this is to write $\mathbb{N} = \bigcup_{k=1}^{\infty} E_k$, where E_k 5 is the set $\{k\}$ containing just the element $k \in \mathbb{N}$. The E_k are disjoint and measurable, so by a result from class,

[18]

$\int_{\mathbb{N}} f\,d\mu = \sum_{k=1}^{\infty} \left(\int_{E_k} f\,d\mu \right)$. 5 But on each E_k, f is simple (it has only one value, $f(k) = \frac{1}{2^k}$ on E_k), so by ~~definition~~ a formula we proved for the 5 integral of a simple function, $\int_{E_k} f\,d\mu = f(k) \cdot \mu(E_k) = \frac{1}{2^k} \cdot 1$.

Therefore $\int_{\mathbb{N}} f\,d\mu = \sum_{k=1}^{\infty} \frac{1}{2^k}$, which is equal to $\underline{1}$. 3

Another way is to write f as the limit of the increasing sequence of simple functions f_n given by $f_n(k) = \begin{cases} \frac{1}{2^k} & \text{for } k \le n \\ 0 & \text{for } k > n. \end{cases}$ 5

Then, since $f_n(k) = \sum_{k=1}^{n} \frac{1}{2^k} \chi_{\{k\}}$, $\int_{\mathbb{N}} f_n\,d\mu = \sum_{k=1}^{n} \frac{1}{2^k} \mu(\{k\})$
$= \sum_{k=1}^{n} \frac{1}{2^k} \cdot 1 = \sum_{k=1}^{n} \frac{1}{2^k}$. 5 By the monotone convergence theorem,

$\int_{\mathbb{N}} f\,d\mu = \lim_{n\to\infty} \int_{\mathbb{N}} f_n\,d\mu = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{2^k} = \sum_{k=1}^{\infty} \frac{1}{2^k} = \underline{1}$. 5 3

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4 a) True: We showed in class that if $\lambda(A) = 0$ then $\int_A f\,d\lambda = 0$ for any non-negative measurable function f,

[6]

so $\mu_f(A) = 0$.

[6] b) False : let $A = [-1,0]$; then $\mu_f(A) = 0$ but $\lambda(A) = 1 \neq 0$.

[6] c) True : let $Z = (-\infty, 0]$; then $\mu_f(Z) = 0$ and for all $A \subseteq \mathbb{R} \sim Z$, $\mu_g(A) = 0$.

[6] d) False : Since $\mu_f \perp \mu_g$ by c), the only way we could have $\mu_f \ll \mu_g$ is if $\mu_f(A) = 0$ for all A, which is not the case ($\mu_f((0,\infty)) = \infty$, for example).
