This is the theorem I was trying to prove on Wednesday, but didn't quite finish. (It's an exercise in Prof. Remling's lecture notes.) The proof that (ii) implies (iii) given here is simpler than the one I came up with on Wednesday.

Theorem. Suppose T is a bounded operator on the Hilbert space H. The following are equivalent:
(i) T is unitary (which means, by definition, that T is a bijection and $\|Tx\| = \|x\|$ for all $x \in H$).
(ii) T is a bijection and

$$\langle Tx, Ty \rangle = \langle x, y \rangle$$

for all $x \in H$ and all $y \in H$.
(iii) $T^*T = TT^* = I$.

Proof. We proved in class that (i) implies (ii) (using the polarization identity) and that (iii) implies (i) (this followed easily from the fact that $\|Tx\| = \langle Tx, Tx \rangle = \langle T^*Tx, x \rangle$.)

To see that (ii) implies (iii), first we show that (ii) implies $T^*T = I$. Indeed, for all x and y in H, we have

$$\begin{aligned} \langle T^*Tx, y \rangle &= \langle Tx, Ty \rangle \quad \text{(by definition of } T^*) \\ &= \langle x, y \rangle \qquad \text{(by (ii))}. \end{aligned}$$

Since this is true for all $y \in H$, it implies that $T^*Tx = x$. And since that is true for all $x \in H$, we have that $T^*T = I$.

Next, we observe that if we apply (ii) with x replaced by $T^{-1}x$ and y replaced by $T^{-1}y$, we get that

$$\langle T(T^{-1}x), T(T^{-1}y) \rangle = \langle T^{-1}x, T^{-1}y \rangle.$$

But this implies that for all x and y in H,

$$\langle x, y \rangle = \langle T^{-1}x, T^{-1}y \rangle.$$

Therefore (ii) holds for T^{-1} as well as for T. So from the preceding paragraph it follows that

$$(T^{-1})^*T^{-1} = I.$$

Now recall that we proved in an earlier proposition that when T is a bijection, then $(T^{-1})^* = (T^*)^{-1}$. So the preceding equation gives

$$(T^*)^{-1}T^{-1} = I.$$

But it is easy to see that whenever A and B are bijections, the composition of their inverses $A^{-1}B^{-1}$ is the same as the inverse of their compositions $(AB)^{-1}$. So from the preceding equation we get

$$(TT^*)^{-1} = I,$$

and hence, of course, $TT^* = I$. So this completes the proof that (ii) implies (iii).

Here's another theorem I mentioned Wednesday, but didn't prove. It turns out we don't need it for the proof of the preceding theorem, but since it is a pretty basic fact, I thought I'd include a proof of it here.

Theorem. If T is a bounded operator on a Hilbert space H, then

$$\overline{R(T^*)} = N(T)^\perp.$$

Proof. We showed in class (see also Theorem 6.2 of Prof. Remling's lecture notes) that $N(T^*) = R(T)^\perp$. Applying this to T^* and using the fact that $T^{**} = T$, we get that $N(T) = R(T^*)^\perp$. Therefore $N(T)^\perp = R(T^*)^{\perp\perp}$. But since $R(T^*)$ is a subspace, it follows from Corollary 5.9 of Prof. Remling's lecture notes (which we also proved in class) that $R(T^*)^{\perp\perp} = \overline{R(T^*)}$. So $N(T)^\perp = \overline{R(T^*)}$.