Definition. We say that a function $f : \mathbf{R} \to \mathbf{R}$ is *additive* if, for all $x \in \mathbf{R}$ and all $y \in \mathbf{R}$,

$$f(x + y) = f(x) + f(y).$$

Lemma 1. Suppose $\phi : \mathbf{R} \to \mathbf{R}$ is additive and suppose ϕ is bounded on $\mathbf{R}$. Then $\phi(x) = 0$ for all $x \in \mathbf{R}$.

Proof. By contradiction. Suppose $\phi(x_0) \neq 0$ for some $x_0 \in \mathbf{R}$. Since ϕ is additive, it follows by induction that $\phi(nx_0) = n\phi(x_0)$ for all $n \in \mathbf{N}$. Since ϕ is bounded, there exists $M \in \mathbf{R}$ such that $|\phi(x)| \leq M$ for all $x \in \mathbf{R}$. Therefore $|\phi(nx_0)| = |n\phi(x_0)| \leq M$ for all $n \in \mathbf{N}$. Hence $n \leq M/|\phi(x_0)|$ for all $n \in \mathbf{N}$. This contradicts the Archimedean property of the real numbers.

Definition. Suppose $\phi : \mathbf{R} \to \mathbf{R}$, and suppose $T > 0$. We say ϕ is *periodic with period* T if, for all $x \in \mathbf{R}$,

$$\phi(x + T) = \phi(x).$$

Lemma 2. Suppose $\phi : \mathbf{R} \to \mathbf{R}$ is periodic with period T, and suppose ϕ is bounded on the interval $[0, T]$. Then ϕ is bounded on all of $\mathbf{R}$.

Proof. This is obvious geometrically: since ϕ is periodic of period T, the graph of ϕ on $\mathbf{R}$ consists of the graph of ϕ on $[0, T]$ repeated over and over on the intervals $[T, 2T]$, $[2T, 3T]$, $[3T, 4T]$, etc., as well as on the intervals $[0, -T]$, $[-2T, -T]$, $[-3T, -2T]$, etc. (Think of the graph of the function $\sin(x)$, which is periodic with period $T = 2\pi$.) If ϕ is bounded on $[0, T]$ by M, it must therefore be bounded by M on all the other intervals as well.

If you want to write a more rigorous proof, you would first show that for every $x \in \mathbf{R}$, there exists an integer m such that $x + mT$ is in $[0, T]$, then use the periodicity of ϕ to deduce that $\phi(x) = \phi(x + mT)$, and so conclude that $\phi(x)$ is bounded by the same number M that bounds ϕ on $[0, T]$. This wouldn't be hard to do, but I imagine you don't feel the need of doing it, since the geometric proof has already convinced you.

Theorem. Suppose $f : \mathbf{R} \to \mathbf{R}$ is additive and suppose there exists some $T > 0$ such that f is bounded on $[0, T]$. Then there exists a number A such that $f(x) = Ax$ for all $x \in \mathbf{R}$.

Proof. Define $A = f(T)/T$, and define

$$\phi(x) = f(x) - Ax$$

for all $x \in \mathbf{R}$. For all $x \in \mathbf{R}$ and all $y \in \mathbf{R}$ we have

$$\phi(x + y) = f(x + y) - A(x + y) = f(x) + f(y) - Ax - Ay = (f(x) - Ax) + (f(y) - Ay) = \phi(x) + \phi(y).$$

This proves that ϕ is additive. Moreover, since $f(x)$ and Ax are both bounded functions on $[0, T]$, then their difference $\phi(x)$ is also bounded on $[0, T]$.

On the other hand, notice that $\phi(T) = f(T) - AT = 0$, by definition of A. Hence, for all $x \in \mathbf{R}$ we have, since ϕ is additive,

$$\phi(x + T) = \phi(x) + \phi(T) = \phi(x) + 0 = \phi(x).$$

This shows that ϕ is periodic on $\mathbf{R}$, and since we already know it is bounded on $[0, T]$, Lemma 2 tells us that ϕ is bounded on all of $\mathbf{R}$. Then Lemma 1 tells us that $\phi(x) = 0$ for all $x \in \mathbf{R}$. From the definition of ϕ it follows that $f(x) - Ax = 0$ for all $x \in \mathbf{R}$, which is what we wanted to prove.

(This proof was taken from the article *The linear functional equation* by G.S. Young, in *The American Mathematical Monthly*, Vol. 65 (1958), pp. 37–38.)