Answers to questions on Exam 1

(3.) Define $h = g - f$, then h is differentiable on $\mathbb{R}$. (2)
Let $x > 0$ be given. (2) By the Mean Value Theorem, there exists $c \in (0, x)$ such that $h(x) - h(0) = h'(c)(x - 0)$. (2)
But $h(0) = g(0) - f(0) = 0 - 0 = 0$, and $h'(c) = g'(c) - f'(c)$, so this implies $h(x) = (g'(c) - f'(c)) \cdot x$ (2). Since $g'(c) \geq f'(c)$, then $g'(c) - f'(c) \geq 0$, and $x > 0$, so $h(x) \geq 0$. (2) Thus $g(x) - f(x) \geq 0$, so $g(x) \geq f(x)$. (2)

(4.) Let $f(x) = 3x$ and $g(x) = e^x + e^{2x} - 2$. (2) Then $f(0) = 0$ and $g(0) = e^0 + e^0 - 2 = 0$. (2) Also $f'(x) = 3$, and
$g'(x) = e^x + 2e^{2x} \geq e^0 + 2e^0 = 3$ (6) (since $x > 0$, and $\frac{d}{dx}(e^x) = e^x > 0$, e^x is increasing, so $e^x > e^0$ and $e^{2x} > e^0$). Therefore, by problem (3.), $f(x) \leq g(x)$ for all $x > 0$. (2)

(5.) Let $x \in \mathbb{R}$ be given. (1) By Taylor's Theorem with $n = 2$, (1) there exists (1) c between x and 0 such that

$$f(x) = f(0) + f'(0) \cdot (x - 0) + \frac{f''(c)}{2}(x - 0)^2.$$

(1) (1) (1) (2)

Since $f(0) = f'(0) = 0$, this implies $f(x) = \frac{f''(c)}{2} \cdot x^2$. (2)
But $f''(c) \geq 0$ by assumption, and $x^2 \geq 0$ (for all $x \in \mathbb{R}$), so $\frac{f''(c)}{2} \cdot x^2 \geq 0$, so $f(x) \geq 0$. (2)

(see also next page) (6.) For all $x \in [a, b]$, we have $-|f(x)| \leq f(x) \leq |f(x)|$. (3)
Let $g(x) = -|f(x)|$ and $h(x) = |f(x)|$. Then g and h are in $R[a,b]$ (this is a fact we haven't proved in class yet, so I should have stated it as an assumption) and $g \leq f \leq h$ on $[a,b]$, so $\int_a^b g \leq \int_a^b f \leq \int_a^b h$. (3)
Hence $\int_a^b (-|f|) \leq \int_a^b f \leq \int_a^b |f|$, so $-\left(\int_a^b |f|\right) \leq \int_a^b f \leq \int_a^b |f|$. (3)
This implies that $\left|\int_a^b f\right| \leq \int_a^b |f|$. (3) (Remember $-Q \leq P \leq Q \Leftrightarrow |P| \leq Q$.)

⑦ Take $[a,b]=[0,1]$, and define $f(x)=\begin{cases}1 & \text{for } 0\le x\le \frac{1}{2}\\ -1 & \text{for } \frac{1}{2}<x\le 1.\end{cases}$ ③

Then f is a step function, so by results proved in class, $f\in\mathcal{R}[0,1]$ ③ and $\int_0^1 f = 1\cdot(\frac{1}{2}-0)+(-1)\cdot(1-\frac{1}{2})=\frac{1}{2}-\frac{1}{2}=0$. ③
However, it is not true that $f(x)=0$ for all $x\in[0,1]$.
This shows that the statement is false. ③

⑧ Let $\varepsilon=1$ ②, and let $\delta>0$ be given. Choose $n\in\mathbb{N}$ such that $\frac{1}{n}<\delta$ ②. Taking $\dot{\mathcal{P}}=\dot{\mathcal{P}}_n$ and $\dot{\mathcal{Q}}=\dot{\mathcal{Q}}_n$ for this n, we have that there exist tagged partitions $\dot{\mathcal{P}}$ and $\dot{\mathcal{Q}}$ of $[a,b]$ such that ② $\|\dot{\mathcal{P}}\|<\frac{1}{n}<\delta$ and $\|\dot{\mathcal{Q}}\|<\frac{1}{n}<\delta$, and $|S(f,\dot{\mathcal{P}})-S(f,\dot{\mathcal{Q}})|\ge\varepsilon$. ② Since $\delta>0$ is arbitrary, this shows that the statement
"for all $\varepsilon>0$, there exists $\delta>0$ s.t. if $\|\dot{\mathcal{P}}\|<\delta$, and $\|\dot{\mathcal{Q}}\|<\delta$, then $|S(f,\dot{\mathcal{P}})-S(f,\dot{\mathcal{Q}})|<\varepsilon$"
is false. ② It then follows from the Cauchy criterion for integrability that $f\notin\mathcal{R}[a,b]$. ②

Alternate proof of ⑥: Let $\varepsilon>0$ be given. Choose $\delta_1>0$ s.t. if $\|\dot{\mathcal{P}}\|<\delta_1$ then $|S(f,\dot{\mathcal{P}})-\int_a^b f|<\frac{\varepsilon}{2}$, and choose $\delta_2>0$ s.t. if $\|\dot{\mathcal{P}}\|<\delta_2$ then $|S(|f|,\dot{\mathcal{P}})-\int_a^b|f||<\frac{\varepsilon}{2}$. ② Let $\dot{\mathcal{P}}$ be any partition of $[a,b]$ s.t. $\|\dot{\mathcal{P}}\|<\min(\delta_1,\delta_2)$. ②

Then $\left|S(|f|,\dot{\mathcal{P}})-\int_a^b|f|\right|<\frac{\varepsilon}{2}\Rightarrow \boxed{S(|f|,\dot{\mathcal{P}})<\int_a^b|f|+\frac{\varepsilon}{2}}$ (1)

and $\left|\int_a^b f\right|-|S(f,\dot{\mathcal{P}})|\le\left|\int_a^b f-S(f,\dot{\mathcal{P}})\right|<\frac{\varepsilon}{2}\Rightarrow$

② $\boxed{\left|\int_a^b f\right|<|S(f,\dot{\mathcal{P}})|+\frac{\varepsilon}{2}.}$ (2)

But $|S(f,\dot{\mathcal{P}})|=\left|\sum_{i=1}^n f(t_i)\cdot(x_i-x_{i-1})\right|\le\sum_{i=1}^n|f(t_i)|(x_i-x_{i-1})=S(|f|,\dot{\mathcal{P}})$ ②

So $\boxed{|S(f,\dot{\mathcal{P}})|\le S(|f|,\dot{\mathcal{P}})}$ (3) Combining (1),(2),(3) gives ② $\left|\int_a^b f\right|<\int_a^b|f|+\varepsilon$

Since $\varepsilon>0$ is arbitrary, this proves $\left|\int_a^b f\right|\le\int_a^b|f|$. ②