Math 4513

1. Show that m^2 is a multiple of 3 only if m is a multiple of 3. Deduce that $\sqrt{3}$ is irrational.

2. Show that if $p = mn$ then $2^p - 1$ is divisible by $2^m - 1$. (Hint: the polynomial $x^n - 1$ can be factored.) Conclude that $2^p - 1$ is prime only if p is prime. Such primes are called *Mersenne primes*. Check that $2^p - 1$ is prime when $p = 2, 3, 5, 7$ but not when $p = 11$.

3. Prove that $2^m + 1$ is prime only if m has no odd divisors; i.e., only if m is a power of 2. (Hint: the polynomial $x^n + 1$ can be factored if n is odd.) Such primes are called *Fermat primes*. The numbers $2^{2^h} + 1$ are prime for $h = 0, 1, 2, 3, 4$, but no other Fermat prime is known.

4. Prove there are infinitely many primes of the form $4n + 3$.

5. Let F_n be the Fibonacci numbers: $F_0 = 0$, $F_1 = 1$, and $F_{n+1} = F_n + F_{n-1}$ for all natural numbers $n \geq 1$.

a. Show that $\gcd(F_n, F_{n+1}) = 1$ for all $n \geq 1$.

b. Show that for all $n \geq 1$,

$$F_{n+2}F_n - F_{n+1}F_{n+1} = -F_{n+1}F_{n-1} + F_nF_n.$$

Deduce that integer solutions x, y of the equation $F_{n+1}x + F_{n-1}y = 1$ are given by $x = F_{n-1}$ and $y = -F_n$ if n is odd, and by $x = -F_{n-1}$ and $y = F_n$ if n is even.

6. Prove the irrationality of the numbers $\sqrt{6}$, $\sqrt[3]{2}$, and $\log_{10} 2$. That is, show that if m and n are natural numbers, then the equations $6 = (m/n)^2$, $2 = (m/n)^3$, and $2 = 10^{(m/n)}$ are impossible.