2.3.3(a) We already explained in class that the heat equation with these boundary conditions has solutions given by

$$u(x,t) = B \sin\left(\frac{n\pi x}{L}\right) e^{-k(n\pi/L)^2 t}$$

where B is an arbitrary constant and $n \in \{1, 2, 3 \ldots\}$. So you can answer this question simply by observing that if we take $B = 6$ and $n = 9$ in this solution, then we get

$$u(x,0) = 6\sin\left(\frac{9\pi x}{L}\right) e^0 = 6\sin\left(\frac{9\pi x}{L}\right),$$

which is the correct initial condition. So the answer is

$$u(x,t) = 6\sin\left(\frac{9\pi x}{L}\right) e^{-k(9\pi/L)^2 t}$$

Alternatively, you could start from the fact, stated in class, that the general solution of the heat equation with the given boundary condition is a linear combination of the above solutions:

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-k(n\pi/L)^2 t},$$

where each B_n is a constant. We also stated in class that in order to achieve the initial condition $u(x,0) = f(x)$, where $f(x)$ is a given function, we should choose the B_n according to the formula

$$B_n = \frac{2}{L}\int_0^L f(\bar{x}) \sin\left(\frac{n\pi\bar{x}}{L}\right)\, d\bar{x},$$

for $n \in \{1, 2, 3 \ldots\}$. Finally, you already know that for all m and n in $\{1, 2, 3 \ldots\}$, we have

$$\int_0^L \sin\left(\frac{m\pi\bar{x}}{L}\right) \sin\left(\frac{n\pi\bar{x}}{L}\right)\, d\bar{x} = \frac{L}{2}\delta_{mn},$$

where δ_{mn} is the Kronecker delta, defined to equal 1 if $m = n$ and 0 if $m \neq n$. Since in this case, we have

$$f(\bar{x}) = 6\sin\left(\frac{9\pi x}{L}\right),$$

we can evaluate the above formula for B_n to get

$$B_n = \frac{2}{L} 6 \frac{L}{2}\delta_{mn} = 6\delta_{9n},$$

or in other words, $B_n = 6$ when $n = 9$ and $B_n = 0$ when $n \neq 9$. Then

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-k(n\pi/L)^2 t} = 6\sin\left(\frac{9\pi x}{L}\right) e^{-k(9\pi/L)^2 t}.$$

This is a much longer way of arriving at the same answer!

2.3.3(d) Here you need to use the formula for B_n, given above, with the function $f(x)$ as specified. This gives

$$\begin{aligned}
B_n &= \frac{2}{L}\int_0^L f(\bar{x}) \sin\left(\frac{n\pi\bar{x}}{L}\right)\, d\bar{x} \\
&= \frac{2}{L}\left(\int_0^{L/2} 1 \cdot \sin\left(\frac{n\pi\bar{x}}{L}\right)\, d\bar{x} + \int_{L/2}^{L} 2 \cdot \sin\left(\frac{n\pi\bar{x}}{L}\right)\, d\bar{x}\right) \\
&= \frac{2}{L}\left(\frac{-L}{n\pi}\left(\cos\left(\frac{n\pi}{2}\right) - 1\right) + \frac{-2L}{n\pi}\left(\cos\left(n\pi\right) - \cos\left(\frac{n\pi}{2}\right)\right)\right) \\
&= \frac{2}{n\pi}\left(-2\cos(n\pi) + \cos\left(\frac{n\pi}{2}\right) + 1\right).
\end{aligned}$$

2.3.5 This is straightforward. To get full credit, though, you should handle both the cases when $m = n$ and when $m \neq n$.

2.4.1(a) You do not need to repeat all the steps of the derivation of the solution of the heat equation with these boundary conditions, because I did them in class, and they are also written out in the text. To get full credit, you only need to mention that the solution is given by

$$u(x,t) = A_0 + \sum_{n=1} \infty A_n \cos\left(\frac{n\pi x}{L}\right) e^{-k(n\pi/L)^2 t},$$

and give the correct formulas for A_0 and A_n ($n \in \{1,2,3,\dots\}$).

For these boundary conditions, the coefficients A_0 and A_n are given in terms of the initial data $f(x)$ by

$$A_0 = \frac{1}{L}\int_0^L f(\bar{x})\ d\bar{x}$$

and, for $n \in \{1,2,3,\dots\}$,

$$A_n = \frac{2}{L}\int_0^L f(\bar{x})\cos\left(\frac{n\pi\bar{x}}{L}\right)\ d\bar{x}.$$

Here we have

$$f(\bar{x}) = \begin{cases} 0 & \bar{x} < L/2 \\ 1 & \bar{x} > L/2 \end{cases},$$

so

$$A_0 = \frac{1}{L}\int_{L/2}^L \ d\bar{x} = 1/2,$$

and, for $n \in \{1,2,3,\dots\}$,

$$A_n = \frac{2}{L}\int_{L/2}^L \cos\left(\frac{n\pi\bar{x}}{L}\right)\ d\bar{x} = \frac{2}{n\pi}\left(\sin(n\pi) - \sin\left(\frac{n\pi}{2}\right)\right) = \left(\frac{-2}{n\pi}\right)\sin\left(\frac{n\pi}{2}\right).$$

2.4.7(b)

Here you have to work out the eigenfunctions for the problem

$$\begin{aligned} \phi''(x) + \lambda\phi(x) &= 0 \\ \phi(0) &= 0 \\ \phi'(L) &= 0, \end{aligned}$$

since this wasn't done in the text. I would like you to be able to actually consider all three of the possibilities $\lambda < 0$, $\lambda = 0$, and $\lambda > 0$ separately, and show carefully that in case $\lambda < 0$ and $\lambda = 0$ there are no non-trivial solutions $\phi(x)$ — so there are no negative eigenvalues, and 0 is not an eigenvalue. (I noted, however, that the problem statement in the text says you can assume $\lambda > 0$, so if we mistakenly took points off your assignment for not doing the cases $\lambda = 0$ and $\lambda < 0$, let me know and I'll restore them.)

You should present a correct derivation of the eigenvalues $\lambda = \left(\frac{n\pi}{2L}\right)^2$, $n \in 1,3,5,7,\dots$, and the corresponding eigenfunctions $\sin\left(\frac{n\pi x}{2L}\right)$.

After you've found the eigenvalues and eigenfunctions, you should say that the solution u is given by

$$u(x,t) = \sum_{n\in\{1,3,5,\dots\}} B_n \sin\left(\frac{n\pi x}{2L}\right) e^{-k(n\pi/(2L))^2 t}$$

(or any equivalent way of writing this sum), and derive the formula for the coefficients

$$B_n = \frac{2}{L} \int_0^L f(\bar{x}) \sin\left(\frac{n\pi\bar{x}}{2L}\right)\, d\bar{x}.$$

You should at least give a little explanation of where the formula comes from: that is, we start from the equation

$$f(x) = u(x,0) = \sum_{n\in\{1,3,5,\dots\}} B_n \sin\left(\frac{n\pi x}{2L}\right),$$

(you can call the eigenfunctions $\phi_n(x)$ instead $\sin\left(\frac{n\pi x}{2L}\right)$ if they like), then multiply both sides of the equation by $\sin\left(\frac{m\pi x}{2L}\right)$ and integrate from $x = 0$ to $x = L$. It was given in the hint to the problem that you can assume that

$$\int_0^L \sin\left(\frac{m\pi x}{2L}\right) \sin\left(\frac{n\pi x}{2L}\right)\, dx = \frac{L}{2}\delta_{mn},$$

so this gives the result

$$\int_0^L f(\bar{x}) \sin\left(\frac{m\pi\bar{x}}{2L}\right)\, d\bar{x} = \sum_{n\in\{1,3,5,\dots\}} B_n \frac{L}{2}\delta_{mn} = \frac{L}{2}B_m,$$

and then multiplying by $2/L$ gives

$$B_m = \frac{2}{L} \int_0^L f(\bar{x}) \sin\left(\frac{m\pi\bar{x}}{2L}\right)\, d\bar{x}.$$

(By the way, I tried to always use $\bar{x}$ as the variable of integration in class so as to not get it mixed up with the variable x in the formula for $u(x,t)$.)