Solution to problem 2.5.1(b)

Substituting the separated solution $u(x,y) = h(x)\phi(y)$ into Laplace's equation gives $\frac{1}{h}\frac{d^2h}{dx^2} = -\frac{1}{\phi}\frac{d^2\phi}{dy^2}$. Both sides of this equation equal a constant, but before choosing a name for this constant we should pause and identify which function we'll be solving the eigenvalue problem for.

For this problem, the homogeneous boundary conditions $\frac{\partial u}{\partial x}(L,y) = 0$, $u(x,0) = 0$, and $u(x,H) = 0$ give us that, for the separated solution to be non-trivial, we must have $\frac{dh}{dx}(L) = 0$, $\phi(0) = 0$, and $\phi(H) = 0$. Since there are two boundary conditions for ϕ and only one for h, the eigenvalue problem will be for ϕ, not h. Therefore, we should choose the name "λ" for the constant which equals both sides of the equation $\frac{1}{h}\frac{d^2h}{dx^2} = -\frac{1}{\phi}\frac{d^2\phi}{dy^2}$, since this will then give

$$\frac{d^2\phi}{dy^2} = -\lambda\phi$$

as the equation for ϕ, and this is the form we are used to for the eigenvalue problem. (If the eigenvalue problem had been for h instead, then a good name for the constant would have been "λ" instead of "$-\lambda$".) We already know from previous work that, for the boundary conditions $\phi(0) = 0$ and $\phi(H) = 0$; the eigenvalues and corresponding eigenfunctions are given by $\lambda = (n\pi/H)^2$ and $\phi(y) = \sin(n\pi y/H)$ for $n = 1, 2, 3, \ldots$).

On the other hand, the equation for h is

$$\frac{d^2h}{dx^2} = \lambda h,$$

and since we now know that $\lambda = (n\pi/H)^2$ $(n = 1, 2, 3, \ldots)$, then from our previous work with this equation we know that $h(x) = B_1 \cosh(n\pi x/H) + B_2 \sinh(n\pi x/H)$. But we also have the condition $\frac{dh}{dx}(L) = 0$ to satisfy. Since $\frac{dh}{dx}(x) = B_1(n\pi/H)\sinh(n\pi x/H) + B_2(n\pi/H)\cosh(n\pi x/H)$, the condition $\frac{dh}{dx}(L) = 0$ tells us that $B_1(n\pi/H)\sinh(n\pi L/H) + B_2(n\pi/H)\cosh(n\pi L/H) = 0$. This equation means that B_1 are B_2 are not independent of each other: once B_1 is given, then B_2 is determined by the equation. Solving for B_2 we see that $B_2 = -B_1\frac{\sinh(n\pi L/H)}{\cosh(n\pi L/H)}$. In other words, $h(x)$ must be given by

$$h(x) = B_1 \cosh(n\pi x/H) + B_2 \sinh(n\pi x/H) = B_1\left(\cosh(n\pi x/H) - \frac{\sinh(n\pi L/H)}{\cosh(n\pi L/H)}\sinh(n\pi x/H)\right).$$

(Note: there is a shorter way to determine the form of $h(x)$, which involves first writing $h(x)$ as $C_1\sinh(n\pi(x-L)/H) + C_2\cosh(n\pi(x-L)/H)$ and then using the condition $\frac{dh}{dx}(L) = 0$ to determine that C_1 must be zero, so $h(x) = C_2\cosh(n\pi(x-L)/H)$. This actually amounts to the same thing as the somewhat longer formula for $h(x)$ given just above.)

To summarize, for this problem, the separated solutions of the PDE and homogeneous boundary conditions are

$$u(x,y) = B\left(\cosh(n\pi x/H) - \frac{\sinh(n\pi L/H)}{\cosh(n\pi L/H)}\sinh(n\pi x/H)\right)\sin(n\pi y/H) \quad (n = 1, 2, 3, \ldots).$$

Now we take

$$u(x,y) = \sum_{n=1}^{\infty} B_n\left(\cosh(n\pi x/H) - \frac{\sinh(n\pi L/H)}{\cosh(n\pi L/H)}\sinh(n\pi x/H)\right)\sin(n\pi y/H),$$

so that

$$\frac{\partial u}{\partial x}(x,y) = \sum_{n=1}^{\infty} B_n\left(\frac{n\pi}{H}\right)\left(\sinh(n\pi x/H) - \frac{\sinh(n\pi L/H)}{\cosh(n\pi L/H)}\cosh(n\pi x/H)\right)\sin(n\pi y/H)$$

and therefore

$$\frac{\partial u}{\partial x}(0,y)=\sum_{n=1}^{\infty}B_n\left(\frac{n\pi}{H}\right)\left(0-\frac{\sinh(n\pi L/H)}{\cosh(n\pi L/H)}\cdot 1\right)\sin(n\pi y/H).$$

We want to satisfy the boundary condition $\frac{\partial u}{\partial x}(0,y)=g(y)$, so we need to have

$$g(y)=\sum_{n=1}^{\infty}B_n\left(\frac{n\pi}{H}\right)\left(\frac{-\sinh(n\pi L/H)}{\cosh(n\pi L/H)}\right)\sin(n\pi y/H).$$

and we have to find $B_0, B_1, B_2, \ldots$ for which this equation is true.

We've already done most of the work needed for this in class: setting $A_n = B_n\left(\frac{n\pi}{H}\right)\left(\frac{-\sinh(n\pi L/H)}{\cosh(n\pi L/H)}\right)$, we have

$$g(y)=\sum_{n=1}^{\infty}A_n\sin(n\pi y/H),$$

so from the first column of the table on page 69, with L replaced by H, x replaced by y, and $f(x)$ replaced by $g(y)$, we get that

$$A_n=\frac{2}{H}\int_0^H g(y)\sin(n\pi y/H)\,dy \quad (n=1,2,3,\ldots).$$

Therefore

$$B_n=\frac{1}{\left(\frac{n\pi}{H}\right)\left(\frac{-\sinh(n\pi L/H)}{\cosh(n\pi L/H)}\right)}\frac{2}{H}\int_0^H g(y)\sin(n\pi y/H)\,dy,$$

which can be simplified a bit to

$$B_n=\frac{-2\cosh(n\pi L/H)}{n\pi\sinh(n\pi L/H)}\int_0^H g(y)\sin(n\pi y/H)\,dy \quad (n=1,2,3,\ldots).$$

So an answer to the problem consists of the formula for $u(x,y)$ given by the sum in the next-to-the-last line on page 1, plus the formula for B_n given above.

This is not the only form the answer to the problem can take. Another way to derive it would be to use the alternate formula for $h(x)$ to write

$$u(x,y)=\sum_{n=1}^{\infty}C_n\cosh(n\pi(x-L)/H)\sin(n\pi y/H),$$

from which, after going through the same steps as above, we can obtain the formula

$$C_n=\frac{-2}{n\pi\sinh(n\pi L/H)}\int_0^H g(y)\sin(n\pi y/H)\,dy \quad (n=1,2,3,\ldots).$$

With a little work one can check that this actually gives the same solution as the more complicated formula above.