Solutions to Problems 2.5.1(a,b,d,e) on Assignment 4

2.5.1(a) We have $u(x, y) = P(x)Q(y)$ where P and Q satisfy the ODEs

$$\frac{d^2P}{dx^2} = -\lambda P \quad \text{and} \quad \frac{d^2Q}{dy^2} = \lambda Q,$$

and the boundary conditions

$$\frac{dP}{dx}(0) = 0, \quad \frac{dP}{dx}(L) = 0, \quad \text{and} \quad Q(0) = 0.$$

If $\lambda > 0$, then as in section 2.4.1 of the text, we find that in order for the ODE and boundary conditions for $P(x)$ to have a non-trivial solution, we must have

$$\lambda = \frac{n^2\pi^2}{L^2}$$

for some integer $n \geq 1$, and in that case the solution is given by

$$P(x) = \cos\left(\frac{n\pi x}{L}\right).$$

Then the equation for $Q(y)$ becomes

$$\frac{d^2Q}{dy^2} = \lambda Q = \frac{n^2\pi^2}{L^2}Q,$$

whose general solution is

$$Q(y) = Ae^{n\pi y/L} + Be^{-n\pi y/L}.$$

Now substituting $y = 0$ on the right-hand side, and using the condition that $Q(0) = 0$ on the left-hand side, we obtain that $0 = A + B$, or $B = -A$. Then substituting into the formula for $Q(y)$ gives

$$Q(y) = Ae^{n\pi y/L} - Ae^{-n\pi y/L},$$

or, what is the same thing,

$$Q(y) = 2A \sinh\left(\frac{n\pi y}{L}\right).$$

Therefore, in the case when $\lambda > 0$, the separated solutions are

$$u(x, y) = \cos\left(\frac{n\pi x}{L}\right) \sinh\left(\frac{n\pi y}{L}\right),$$

where n can be any positive whole number.

If $\lambda = 0$, then again as in section 2.4.1, we find that the ODE and boundary conditions for $P(x)$ do have a non-trivial solution, so $\lambda = 0$ is an eigenvalue, and a solution is $P(x) = 1$ (or any other constant). Now the equation for $Q(y)$ becomes

$$\frac{d^2Q}{dy^2} = \lambda Q = 0,$$

whose general solution is

$$Q(y) = Ay + B.$$

Substituting $y = 0$ on the right-hand side, and using the condition that $Q(0) = 0$ on the left-hand side, we obtain that $0 = B$. Hence $Q(y) = Ay$. Therefore, in the case when $\lambda = 0$, the separated solution is

$$u(x, y) = y.$$

Finally we have to satisfy the inhomogeneous boundary condition $u(x, H) = f(x)$, where $f(x)$ is assumed to be a known (given) function. First we form a superposition of separated solutions:

$$u(x,y) = A_0 y + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) \sinh\left(\frac{n\pi y}{L}\right).$$

Setting $y = H$ on the right-hand side and using the condition $u(x, H) = f(x)$ on the left-hand side, we get

$$f(x) = A_0 H + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) \sinh\left(\frac{n\pi H}{L}\right),$$

an equation which is supposed to hold for all x in the interval $0 \leq x \leq L$.

Now, to determine the constants $A_0, A_1, A_2, \ldots$, we multiply both sides of the preceding equation by each of the functions $1, \cos\left(\frac{\pi x}{L}\right), \cos\left(\frac{2\pi x}{L}\right), \ldots$, and integrate from $x = 0$ to $x = L$. Multiplying by 1 and integrating (using $\tilde{x}$ as the variable of integration, so as not to confuse it with the variable x which appears in the formula for $u(x, y)$), we get the equation

$$\int_0^L f(\tilde{x})\, d\tilde{x} = \int_0^L A_0 H \, d\tilde{x},$$

since the integrals of all the terms in the infinite sum are zero. Performing the integral on the right side and solving for A_0 gives

$$A_0 = \frac{1}{LH} \int_0^L f(\tilde{x})\, d\tilde{x}.$$

Similarly, multiplying the equation for $f(x)$ by $\cos\left(\frac{m\pi x}{L}\right)$ and using the orthogonality of $\cos\left(\frac{m\pi x}{L}\right)$ to all the other functions $\cos\left(\frac{n\pi x}{L}\right)$ with $n \neq m$, we get after integrating from $x = 0$ to $x = L$ that

$$\int_0^L f(\tilde{x}) \cos\left(\frac{m\pi \tilde{x}}{L}\right) d\tilde{x} = A_m \int_0^L \left(\cos\left(\frac{m\pi x}{L}\right)\right)^2 d\tilde{x} = A_m \frac{L}{2},$$

so

$$A_m = \frac{2}{L} \int_0^L f(\tilde{x}) \cos\left(\frac{m\pi \tilde{x}}{L}\right) d\tilde{x},$$

for any $m \geq 1$.

2.5.1(b) Here $u(x, y) = P(x)Q(y)$ where

$$\frac{d^2 P}{dx^2} = \lambda P, \quad \frac{d^2 Q}{dy^2} = -\lambda Q,$$

and

$$\frac{dP}{dx}(L) = 0, \quad Q(0) = 0, \quad \text{and} \quad Q(H) = 0.$$

(We decided to put λ in the equation for P and $-\lambda$ in the equation for Q, not because we had to do it, but just because it means having to write fewer minus signs later on.) The problem for $Q(y)$, $y \in [0, H]$, is a Sturm-Liouville problem, because there are two homogeneous boundary conditions. We have solved this problem before, and obtained that non-trivial solutions only exist for $\lambda = (n\pi/H)^2$, $n = 1, 2, 3, \ldots$, and are given by

$$Q(y) = \sin\left(\frac{n\pi y}{H}\right),$$

For these values of λ, the equation for $P(x)$ becomes

$$\frac{d^2 P}{dx^2} = \left(\frac{n\pi}{H}\right)^2 P,$$

and we can either write the general solution as

$$P(x) = Ae^{n\pi x/H} + Be^{-n\pi x/H},$$

or, what is more convenient for this problem, as

$$P(x) = C\cosh\left(\frac{n\pi(x-L)}{H}\right) + D\sinh\left(\frac{n\pi(x-L)}{H}\right).$$

Differentiating the latter equation gives

$$\frac{dP}{dx} = C\left(\frac{n\pi}{H}\right)\sinh\left(\frac{n\pi(x-L)}{H}\right) + D\left(\frac{n\pi}{H}\right)\cosh\left(\frac{n\pi(x-L)}{H}\right),$$

and substituting $x = L$ and using that $\frac{dP}{dx} = 0$ when $x = L$, we get $0 = D(n\pi/H)$, so $D = 0$. Therefore

$$P(x) = \cosh\left(\frac{n\pi(x-L)}{H}\right).$$

To satisfy the inhomogeneous condition $\frac{\partial u}{\partial x}(0, y) = g(y)$, we put

$$u(x, y) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi y}{H}\right)\cosh\left(\frac{n\pi(x-L)}{H}\right),$$

and take the derivative with respect to x to give

$$\frac{\partial u}{\partial x} = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi y}{H}\right)\left(\frac{n\pi}{H}\right)\sinh\left(\frac{n\pi(x-L)}{H}\right).$$

Now putting $x = 0$ on the right and using $\frac{\partial u}{\partial x}(0, y) = g(y)$ on the left, we get

$$g(y) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi y}{H}\right)\left(\frac{n\pi}{H}\right)\sinh\left(\frac{-n\pi L}{H}\right).$$

It follows by the usual procedure (multiplying both sides by $\sin\left(\frac{m\pi y}{H}\right)$ and integrating from $y = 0$ to $y = H$) that for each $m \geq 1$,

$$B_m = \frac{H}{m\pi \sinh(-m\pi L/H)} \int_0^H g(\tilde{y}) \sin\left(\frac{m\pi\tilde{y}}{H}\right)\, d\tilde{y}.$$

2.5.1(d) Here $u(x, y) = P(x)Q(y)$ where

$$\frac{d^2P}{dx^2} = \lambda P, \quad \frac{d^2Q}{dy^2} = -\lambda Q,$$

and

$$P(L) = 0, \quad \frac{dQ}{dy}(0) = 0, \quad \text{and} \quad Q(H) = 0.$$

The Sturm-Liouville problem for $Q(y)$ was solved in the exercises for Chapter 2 (see problem 2.3.2(e)). It has non-trivial solutions only when $\sqrt{\lambda}\, H$ is an odd multiple of $\pi/2$; i.e., only for $\lambda = \left(\frac{(2n-1)\pi}{2H}\right)^2$, where $n = 1, 2, 3, \ldots$, and the solutions are given by

$$Q(y) = \cos\left(\frac{(2n-1)\pi y}{2H}\right).$$

For each such value of λ, the corresponding general solution for the equation for $P(x)$ is then

$$P(x) = A\cosh\left(\frac{(2n-1)\pi(x-L)}{2H}\right) + B\sinh\left(\frac{(2n-1)\pi(x-L)}{2H}\right),$$

and the condition $P(L) = 0$ gives us that $A = 0$. So

$$P(x) = B\sinh\left(\frac{(2n-1)\pi(x-L)}{2H}\right).$$

Now put

$$u(x,y) = \sum_{n=1}^{\infty} A_n \cos\left(\frac{(2n-1)\pi y}{2H}\right)\sinh\left(\frac{(2n-1)\pi(x-L)}{2H}\right).$$

Letting $x = 0$ and using the boundary condition $u(0,y) = g(y)$, we get

$$g(y) = \sum_{n=1}^{\infty} A_n \cos\left(\frac{(2n-1)\pi y}{2H}\right)\sinh\left(\frac{(2n-1)\pi(-L)}{2H}\right),$$

so by the usual method we find that for all $m \geq 1$,

$$A_m = \frac{2}{H\sinh\left(\frac{(2m-1)\pi(-L)}{2H}\right)}\int_0^H g(\tilde{y})\cos\left(\frac{(2m-1)\pi\tilde{y}}{2H}\right)\,d\tilde{y}.$$

2.5.1(e) Here $u(x,y) = P(x)Q(y)$ where

$$\frac{d^2P}{dx^2} = \lambda P, \quad \frac{d^2Q}{dy^2} = -\lambda Q,$$

and

$$P(0) = 0, \quad P(L) = 0, \quad \text{and} \quad Q(0) - \frac{dQ}{dy}(0) = 0.$$

The Sturm-Liouville problem for $P(x)$ has non-trivial solutions only when $\lambda = \left(\frac{n\pi}{L}\right)^2$, where $n = 1, 2, 3, \ldots$, and the solutions are given by

$$P(x) = \sin\left(\frac{n\pi x}{L}\right).$$

For each such value of λ, the corresponding general solution for the equation for $Q(y)$ is then

$$Q(y) = A\cosh\left(\frac{n\pi y}{L}\right) + B\sinh\left(\frac{n\pi y}{L}\right).$$

Taking the derivative with respect to y gives

$$\frac{dQ}{dy} = A\left(\frac{n\pi}{L}\right)\sinh\left(\frac{n\pi y}{L}\right) + B\left(\frac{n\pi}{L}\right)\cosh\left(\frac{n\pi y}{L}\right).$$

Now substituting $y = 0$ into the last two equations and subtracting gives

$$Q(0) - \frac{dQ}{dy}(0) = A - B\left(\frac{n\pi}{L}\right),$$

so the condition that $Q(0) - \frac{dQ}{dy}(0) = 0$ gives us that

$$A = B\left(\frac{n\pi}{L}\right).$$

We can then substitute this expression for A into the formula for $Q(y)$ to get

$$Q(y) = B\left(\frac{n\pi}{L}\right)\cosh\left(\frac{n\pi y}{L}\right) + B\sinh\left(\frac{n\pi y}{L}\right).$$

Now put

$$u(x,y) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)\left[\left(\frac{n\pi}{L}\right)\cosh\left(\frac{n\pi y}{L}\right) + \sinh\left(\frac{n\pi y}{L}\right)\right].$$

Letting $y = H$ and using the boundary condition $u(x,H) = f(x)$, we get

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)\left[\left(\frac{n\pi}{L}\right)\cosh\left(\frac{n\pi H}{L}\right) + \sinh\left(\frac{n\pi H}{L}\right)\right],$$

so by the usual method we find that for all $m \geq 1$,

$$B_m = \frac{2}{H\left[\left(\frac{m\pi}{L}\right)\cosh\left(\frac{m\pi H}{L}\right) + \sinh\left(\frac{m\pi H}{L}\right)\right]} \int_0^H f(\tilde{x}) \sin\left(\frac{m\pi\tilde{x}}{L}\right)\, d\tilde{x}.$$