2.5.16 - Solution

(a) Let R stand for the rectangular region in the plane given by $0 \le x \le L$ and $0 \le y \le H$. Let C stand for the four line segments which form the boundary of this region.

According to the law of conservation of heat energy (equation (1.5.1) in the text), if there are no sources ($Q \equiv 0$) then the temperature u within R and the heat flux vector $\vec{\phi}$ on C satisfy

$$(1) \qquad \frac{d}{dt} \iint_R c\rho u \, dV = \oint_C \vec{\phi} \cdot \vec{n} \, d\tau .$$

Here c is the specific heat, ρ is the density, and $\vec{n}$ is the outward unit vector normal to C. Since this is a two-dimensional problem, the triple integral in (1.5.1) is replaced by a double integral, and the surface integral in (1.5.1) is replaced by a line integral over C, as explained on p. 27 of the text.

In this problem, the temperature is at equilibrium, so $\frac{\partial u}{\partial t} \equiv 0$, and hence $\frac{d}{dt} \iint_R c\rho u \, dV = \iint_R c\rho \frac{\partial u}{\partial t} \, dV = 0$.

Therefore, equation (1) gives

$$(2) \qquad \oint_C \vec{\phi} \cdot \vec{n} \, d\tau = 0 .$$

Now by Fourier's law of heat conduction, equation (1.5.7) in the text, we have $\vec{\phi} = -K_0 \vec{\nabla} u$, where $-K_0$

is the thermal conductivity. Therefore from (2) we get

$$(3) \qquad \oint_C (\vec{\nabla} u)\cdot \vec{n} = 0 .$$

The line integral in (3) splits into four integrals, one along each of the line segments forming the boundary of R. On the right boundary, $\vec{n} = \vec{i}$, so $\vec{\nabla} u \cdot \vec{n} = \left(\frac{\partial u}{\partial x}\vec{i} + \frac{\partial u}{\partial y}\vec{j}\right)\cdot \vec{i} = \frac{\partial u}{\partial x}$. On the left boundary, $\vec{n} = -\vec{i}$, so $\vec{\nabla} u \cdot \vec{n} = -\frac{\partial u}{\partial x}$.

On the top boundary, $\vec{n} = \vec{j}$, so $\vec{\nabla} u \cdot \vec{n} = \frac{\partial u}{\partial y}$. On the bottom boundary, $\vec{n} = -\vec{j}$, so $\vec{\nabla} u \cdot \vec{n} = -\frac{\partial u}{\partial y}$. Therefore, equation (3) becomes:

$$0 = \underbrace{\int_0^L -\frac{\partial u}{\partial y}(x,0)\,dx}_{\text{(bottom)}} + \underbrace{\int_0^H \frac{\partial u}{\partial x}(L,y)\,dy}_{\text{(right)}} + \underbrace{\int_0^L \frac{\partial u}{\partial y}(x,H)\,dx}_{\text{(top)}} + \underbrace{\int_0^H -\frac{\partial u}{\partial x}(0,y)\,dy}_{\text{(left)}}$$

From the given boundary conditions, $\frac{\partial u}{\partial y}(x,0) \equiv 0$, $\frac{\partial u}{\partial x}(L,y) = g(y)$, $\frac{\partial u}{\partial y}(x,H) = f(x)$, and $\frac{\partial u}{\partial x}(0,y) \equiv 0$, so we get

$$\boxed{(4) \quad 0 = \int_0^H g(y)\,dy + \int_0^L f(x)\,dx ,}$$

which is our solvability condition.

Remember that (4) is the same equation as (2), so the physical interpretation of (4) is the same as that of (2): namely that the total amount of heat flowing through the boundary C (per unit time) is 0.

In other words, for the temperature within R to be at equilibrium, there must be no accumulation or loss of heat energy into or out of R.

Note: The same result can be deduced mathematically from the equation $\nabla^2 u = 0$ on R, using the divergence theorem. See equation (2.5.61) on page 85.

(b) Suppose $f(x)$ and $g(y)$ are constants, say $f(x) \equiv B_1$ and $g(x) \equiv B_2$. The condition of part (a) is that

$$\int_0^H g(y)\,dy + \int_0^L f(x)\,dx = 0, \text{ or } \int_0^H B_2\,dy + \int_0^L B_1\,dx = 0,$$

or $B_2 H + B_1 L = 0$. Thus $\frac{-B_2}{L} = \frac{B_1}{H}$.

Now let $u(x,y) = A(x^2 - y^2)$. We see that $\nabla^2 u =$

$$= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2A + (-2A) = 0,$$ so u satisfies Laplace's

equation. We also have to check the four boundary conditions: We have:

$$\frac{\partial u}{\partial x}(0,y) = 2Ax\Big|_{x=0} = 0,$$

$$\frac{\partial u}{\partial y}(x,0) = -2Ay\Big|_{y=0} = 0,$$

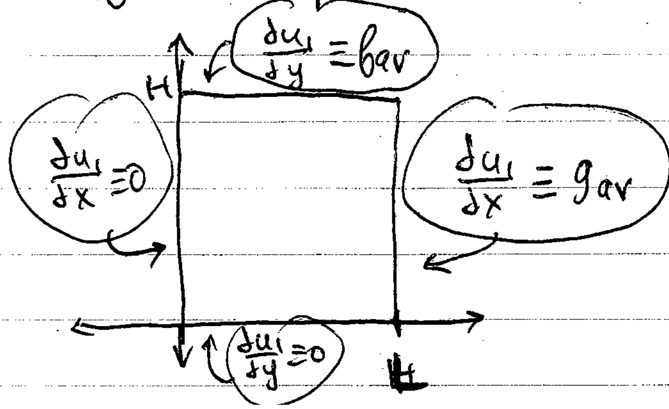
$$\frac{\partial u}{\partial x}(L,y) = 2Ax\Big|_{x=L} = 2AL, \quad \text{and}$$

$$\frac{\partial u}{\partial y}(x, H) = -2Ay\Big|_{y=H} = -2AH.$$

In order to satisfy the boundary conditions $\frac{\partial u}{\partial x}(L,y) = g(y)$ and $\frac{\partial u}{\partial y}(x,H) = f(x)$, we see from the last two equations that we have to choose the constant A so that both the equations $2AL = g(y) \equiv B_2$ and $-2AH = f(x) \equiv B_1$ hold. In other words, we must take $A = \frac{B_2}{2L}$ and $A = \frac{-B_1}{2H}$. This would be impossible if $\frac{B_2}{2L}$ and $\frac{-B_1}{2H}$ were different constants, but as noted above, they are the same number because of the condition of part (a).

(c) Let $f_{av} = \frac{1}{L}\int_0^L f(x)\,dx$ and $g_{av} = \frac{1}{H}\int_0^H g(y)\,dy$.

Notice that f_{av} and g_{av} are constants. By part (b), the solution u_1 of the problem $\nabla^2 u_1 = 0$, with boundary conditions as shown in the diagram at left, will exist and will be given by



(4½) $\qquad u_1 = A(x^2 - y^2)$

(for $A = (B_2/2L) = (-B_1/2H)$), provided the condition

(5) $\qquad B_2 H + B_1 L = 0$

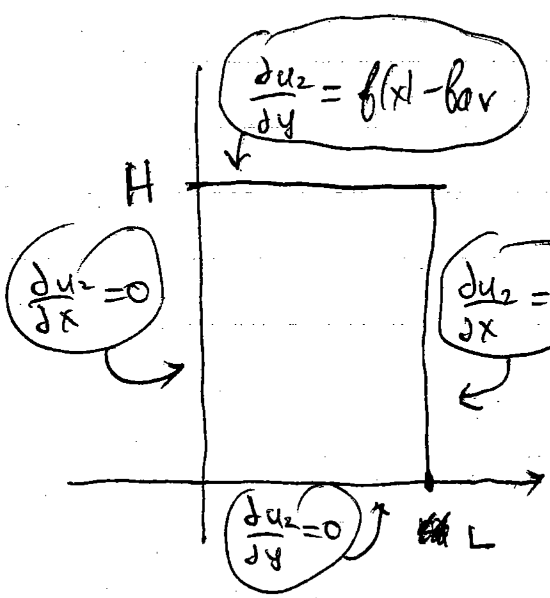
is satisfied, where here $B_2 = g_{av}$ and $B_1 = f_{av}$.

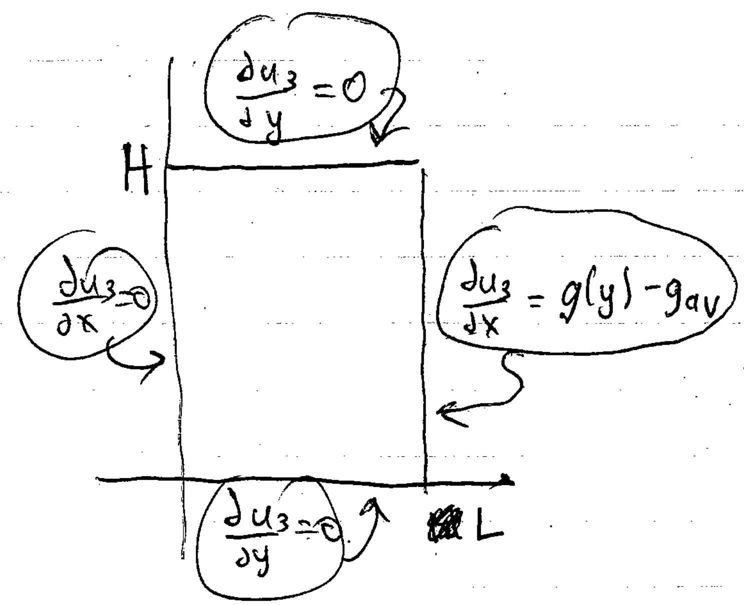
But we are given the condition $0 = \int_0^H g(y)\,dy + \int_0^L f(x)\,dx$,

which tells us that $0 = H \cdot \frac{1}{H} \int_0^H g(y)\,dy + L \cdot \frac{1}{L} \int_0^L f(x)\,dx$, or

$0 = H \cdot g_{av} + L \cdot f_{av}$, or $0 = B_2 H + B_1 L$. So (5) is satisfied, and the solution u_1 is valid.

Now let u_2 and u_3 be the solutions of $\nabla^2 u_2 = 0$ and $\nabla^2 u_3 = 0$ with the boundary conditions shown in the diagrams below:





Observe that in the problem for u_2, because we have subtracted the constant f_{av} from $f(x)$, it is now true that $\int_0^L \frac{\partial u_2}{\partial y}(x,H)\,dx = 0$:

$$\int_0^L \frac{\partial u_2}{\partial y}(x,H)\,dx = \int_0^L [f(x) - f_{av}]\,dx = \int_0^L f(x)\,dx - \int_0^L f_{av}\,dx =$$

$$= \int_0^L f(x)\,dx - f_{av}\cdot L =$$

$$= \int_0^L f(x)\,dx - \left(\frac{1}{L}\int_0^L f(x)\,dx\right)\cdot L =$$

$$= \int_0^L f(x)\,dx - \int_0^L f(x)\,dx = 0.$$

Therefore, from exercise $\S 2.5.2 \S$ on page 85, we know that the problem for u_2 has a solution, and the solution is given by

(6) $$u_2(x,y) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right)\cosh\left(\frac{n\pi y}{L}\right),$$

where A_0 is arbitrary (not determined by the given conditions), and for $n \geq 1$,

$$A_n = \frac{L}{n\pi}\cdot\frac{1}{\sinh\left(\frac{n\pi H}{L}\right)}\cdot\frac{2}{L}\int_0^L \{f(w) - f_{av}\}\cos\left(\frac{n\pi w}{L}\right)dw.$$

In fact, since $\int_0^L f_{av}\cos\left(\frac{n\pi w}{L}\right)dw = f_{av}\int_0^L \cos\left(\frac{n\pi w}{L}\right)dw = 0$ for $n \geq 1$, we can rewrite the formula for A_n as

(7) $$A_n = \frac{2}{n\pi\sinh\left(\frac{n\pi H}{L}\right)}\int_0^L f(w)\cos\left(\frac{n\pi w}{L}\right)dw.$$

In the problem for u_3, the only difference from the problem for u_2 is that the variables x and y have been switched (with L replaced by H and f by g), so we get

$$(8) \quad u_3(x,y) = B_0 + \sum_{n=1}^{\infty} B_n \cos\left(\frac{n\pi y}{H}\right) \cosh\left(\frac{n\pi x}{H}\right),$$

where B_0 is arbitrary and, for $n \geq 1$,

$$(9) \quad B_n = \frac{2}{n\pi \sinh\left(\frac{n\pi L}{H}\right)} \int_0^H g(w) \cos\left(\frac{n\pi w}{H}\right) dw .$$

Finally, we observe that the solution to the problem for u is given by $u = u_1 + u_2 + u_3$ (to see this, notice that

$$\frac{\partial u}{\partial y}(x,H) = f(x) = f_{av} + (f(x) - f_{av}) + 0$$
$$= \frac{\partial u_1}{\partial y}(x,H) + \frac{\partial u_2}{\partial y}(x,H) + \frac{\partial u_3}{\partial y}(x,H);$$

and similarly for the other three boundary conditions).
Combining the formulas for u_1, u_2, u_3 in $(4\frac{1}{2})$, (6), and (8), we get

$$\boxed{u = A(x^2 - y^2) + C_0 + \sum_{n=1}^{\infty} \left\{ A_n \cos\left(\frac{n\pi x}{L}\right) \cosh\left(\frac{n\pi y}{L}\right) + B_n \cos\left(\frac{n\pi y}{H}\right) \cosh\left(\frac{n\pi x}{H}\right) \right\}}$$

where $A = \frac{g_{av}}{2L} = -\frac{f_{av}}{2H} = \frac{1}{2LH} \int_0^H g(w)\,dw = \frac{-1}{2HL} \int_0^L f(w)\,dw$,

$C_0 = A_0 + B_0$ is an arbitrary constant, and A_n and B_n are given by the equations (7) and (9) for all $n \geq 1$.